

QUIVER-THEORETICAL APPROACH TO DYNAMICAL YANG-BAXTER MAPS

DIOGO KENDY MATSUMOTO

*Center for Promotion of Educational Innovation
Shibaura Institute of Technology
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.*

KENICHI SHIMIZU

*Department of Mathematical Sciences
Shibaura Institute of Technology
307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan.*

ABSTRACT. A dynamical Yang-Baxter map, introduced by Shibukawa, is a solution of the set-theoretical analogue of the dynamical Yang-Baxter equation. In this paper, we initiate a quiver-theoretical approach for the study of dynamical Yang-Baxter maps. Our key observation is that the category of dynamical sets over a set Λ , introduced by Shibukawa to establish a categorical framework to deal with dynamical Yang-Baxter maps, can be embedded into the category of quivers with vertices Λ . By using this embedding, we shed light on Shibukawa's classification result of a certain class of dynamical Yang-Baxter maps and extend his construction to obtain a new class of dynamical Yang-Baxter maps. We also discuss a relation between Shibukawa's bialgebroid associated to a dynamical Yang-Baxter map and Hayashi's weak bialgebra associated to a star-triangular face model.

1. INTRODUCTION

The Yang-Baxter equation was first considered independently by McGuire [16] and Yang [26] in their study of one-dimensional many body problems. Finding a (constant) solution of the equation is equivalent to solving the equation

$$(1.1) \quad (\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\text{id}_V \otimes \sigma) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)$$

for a linear operator $\sigma : V \otimes V \rightarrow V \otimes V$ on the two-fold tensor product of a vector space V . Thus, abusing terminology, we refer to (1.1) as the Yang-Baxter equation in this paper. As a natural analogue of (1.1), Drinfeld [3] proposed to investigate the set-theoretical Yang-Baxter equation

$$(1.2) \quad (\sigma \times \text{id}_X)(\text{id}_X \times \sigma)(\text{id}_X \times \sigma) = (\text{id}_X \times \sigma)(\sigma \times \text{id}_X)(\text{id}_X \times \sigma)$$

E-mail addresses: diogo-sw@shibaura-it.ac.jp, kshimizu@shibaura-it.ac.jp.

The second author (K.S.) is supported by JSPS KAKENHI Grant Number JP16K17568.

for a map $\sigma : X \times X \rightarrow X \times X$ on the two-fold Cartesian product of a set X . Solutions of (1.2), called *Yang-Baxter maps*, have been studied as well as the solutions of the ordinary Yang-Baxter equation.

Gervais and Neveu [7] have introduced a generalization of the Yang-Baxter equation (1.2), called the *dynamical Yang-Baxter equation*. Let \mathfrak{h} be a commutative Lie algebra, and let V be a diagonalizable \mathfrak{h} -module. The dynamical Yang-Baxter equation on V is (equivalent to) the equation

$$(1.3) \quad \sigma(\lambda)_{12} \circ \sigma(\lambda - h^{(1)})_{23} \circ \sigma(\lambda)_{12} = \sigma(\lambda - h^{(1)})_{23} \circ \sigma(\lambda)_{12} \circ \sigma(\lambda - h^{(1)})_{23}$$

for a family $\{\sigma(\lambda) \in \text{End}_{\mathfrak{h}}(V \otimes V)\}_{\lambda \in \mathfrak{h}^*}$ of \mathfrak{h} -equivariant linear operators. Here, $\sigma(\lambda)_{12}$ and $\sigma(\lambda - h^{(1)})_{23}$ are linear operators on $V \otimes V \otimes V$ defined by

$$\begin{aligned} \sigma(\lambda)_{12}(v_1 \otimes v_2 \otimes v_3) &= \sigma(\lambda)(v_1 \otimes v_2) \otimes v_3, \\ \sigma(\lambda - h^{(1)})_{23}(v_1 \otimes v_2 \otimes v_3) &= v_1 \otimes \sigma(\lambda - \text{wt}(v_1))(v_2 \otimes v_3) \end{aligned}$$

for $\lambda \in \mathfrak{h}$ and weight vectors $v_1, v_2, v_3 \in V$. Felder [6] studied mathematical aspects of this equation. Motivated by its relation to conformal field theory and statistical mechanics, the dynamical Yang-Baxter equation also have been studied extensively; see, *e.g.*, the lecture book of Etingof-Latour [4] and references therein.

It is interesting to study a set-theoretical analogue — like (1.2) — of the dynamical Yang-Baxter equation (1.3). Shibukawa gave a mathematical formulation of such an equation: Let Λ be a non-empty set, and let X be a set equipped with a map $\Lambda \times X \rightarrow \Lambda$ expressed as $(\lambda, x) \mapsto \lambda \triangleleft x$. The *set-theoretical dynamical Yang-Baxter equation* [19, 20, 21] is the equation

$$(1.4) \quad \sigma(\lambda)_{12} \circ \sigma(\lambda \triangleleft X^{(1)})_{23} \circ \sigma(\lambda)_{12} = \sigma(\lambda \triangleleft X^{(1)})_{23} \circ \sigma(\lambda)_{12} \circ \sigma(\lambda \triangleleft X^{(1)})_{23}$$

for a family $\{\sigma(\lambda) : X \times X \rightarrow X \times X\}_{\lambda \in \Lambda}$ of maps parametrized by Λ . Here, $\sigma(\lambda)_{12}$ and $\sigma(\lambda \triangleleft X^{(1)})_{23}$ are maps on $X \times X \times X$ defined by

$$\begin{aligned} \sigma(\lambda)_{12}(x, y, z) &= (\sigma(\lambda)(x, y), z), \\ \sigma(\lambda \triangleleft X^{(1)})_{23}(x, y, z) &= (x, \sigma(\lambda \triangleleft x)(y, z)) \end{aligned}$$

for $\lambda \in \Lambda$ and $x, y, z \in X$. We usually require $\sigma(\lambda)$ to satisfy the condition

$$(1.5) \quad (\lambda \triangleleft (x \xrightarrow{\lambda} y)) \triangleleft (x \xleftarrow{\lambda} y) = (\lambda \triangleleft x) \triangleleft y \quad \text{for all } x, y \in X \text{ and } \lambda \in \Lambda,$$

where the symbols $x \xrightarrow{\lambda} y$ and $x \xleftarrow{\lambda} y$ are defined by

$$(1.6) \quad \sigma(\lambda)(x, y) = (x \xrightarrow{\lambda} y, x \xleftarrow{\lambda} y)$$

for $\lambda \in \Lambda$ and $x, y \in X$. The condition (1.5) corresponds to the \mathfrak{h} -equivariance property of $\sigma(\lambda)$ in (1.3) and thus it is called the *invariance condition* [21] or the *weight zero condition* [17].

Definition 1.1 (Shibukawa [20, 21]). A solution $\{\sigma(\lambda) : X \times X \rightarrow X \times X\}_{\lambda \in \Lambda}$ of the set-theoretical dynamical Yang-Baxter equation (1.4) satisfying the invariance condition (1.5) is called a *dynamical Yang-Baxter map* on X .

Unlike the original dynamical Yang-Baxter equation, it is not known whether dynamical Yang-Baxter maps relate to physical models. At the moment, dynamical Yang-Baxter maps are studied purely from the viewpoint of algebra and combinatorics [9, 14, 15, 17, 19, 20, 21, 22, 23].

In this paper, we give a new and systematic method to study dynamical Yang-Baxter maps. Shibukawa [19] has introduced the monoidal category \mathbf{DSet}_Λ as a useful framework to deal with dynamical Yang-Baxter maps. As Shibukawa pointed out in [19], a dynamical Yang-Baxter map is just a braided object of \mathbf{DSet}_Λ . Our key observation is that the category \mathbf{DSet}_Λ can be fully embedded into the monoidal category \mathbf{Quiv}_Λ of quivers with vertices Λ (Theorem 2.7). Thus, through the embedding, a dynamical Yang-Baxter map gives rise to a braided quiver studied by Andruskiewitsch [1]. Following this scheme, one can utilize quiver-theoretical methods to the study of dynamical Yang-Baxter maps.

Organization of this paper. We describe the organization of this paper. In Section 2, we first introduce the monoidal categories \mathbf{DSet}_Λ and \mathbf{Quiv}_Λ for a non-empty set Λ . We then construct a fully faithful strong monoidal functor $Q : \mathbf{DSet}_\Lambda \rightarrow \mathbf{Quiv}_\Lambda$ (Theorem 2.7). This functor turns a dynamical Yang-Baxter map into a braided quiver, *i.e.*, a braided object of \mathbf{Quiv}_Λ (Theorem 2.9).

In Section 3, we reexamine Shibukawa's classification result (cited as Theorem 3.3 of this paper) on the dynamical Yang-Baxter maps on a dynamical set of a certain form, which we call a dynamical set of *PH type* (Definition 3.1). We give a different proof of the classification result from the viewpoint of our quiver-theoretical approach. We also classify the dynamical Yang-Baxter maps on a dynamical set of PH type up to equivalence (Theorem 3.4).

In Section 4, we consider a class of dynamical sets larger than the class of dynamical sets of PH type. We do not give a classification of the dynamical Yang-Baxter maps on such a dynamical set, but invent a new class of Yang-Baxter maps again by our quiver-theoretical approach.

In Section 5, we concern two constructions of a weak bialgebra from a dynamical Yang-Baxter map (X, σ) satisfying a certain condition. On the one hand, Shibukawa [22] constructed a weak bialgebra $\mathfrak{B}(\sigma)$ from such a dynamical Yang-Baxter map (X, σ) . On the other hand, we obtain another weak bialgebra $\mathfrak{A}(w_\sigma)$ by applying Hayashi's construction [8] to the linearization of the braided quiver associated to (X, σ) . We give a natural weak bialgebra map $\phi : \mathfrak{A}(w_\sigma) \rightarrow \mathfrak{B}(\sigma)$ which is not an isomorphism in general (Theorem 5.4 and Remark 5.5).

Acknowledgment. We are grateful to Takahiro Hayashi and Youichi Shibukawa for helpful discussion. The second author (K.S.) is supported by JSPS KAKENHI Grant Number JP16K17568.

2. DYNAMICAL SETS AND QUIVERS

2.1. Monoidal categories and functors. We refer the reader to Mac Lane [13] for the basic notions in the category theory. A *monoidal category* [13, VII.1] is a category \mathcal{C} endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (called the tensor product), an object $\mathbf{1} \in \mathcal{C}$ (called the unit object), and natural isomorphisms

$$(2.1) \quad (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \quad \text{and} \quad \mathbf{1} \otimes X \cong X \cong X \otimes \mathbf{1} \quad (X, Y, Z \in \mathcal{C})$$

obeying the pentagon and the triangle axioms. A monoidal category is said to be *strict* if the natural isomorphisms (2.1) are the identities. In view of the Mac Lane coherence theorem, we may assume that all monoidal categories are strict when we discuss the general theory of monoidal categories.

Let \mathcal{C} and \mathcal{D} be (strict) monoidal categories. A *monoidal functor* [13, XI.2] from \mathcal{C} to \mathcal{D} is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ endowed with a natural transformation

$$F_{X,Y}^{(2)} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y) \quad (X, Y \in \mathcal{C})$$

and a morphism $F^{(0)} : \mathbf{1} \rightarrow F(\mathbf{1})$ satisfying the equations

$$(2.2) \quad F_{X \otimes Y, Z}^{(2)} \circ (F_{X,Y}^{(2)} \otimes \text{id}_{F(Z)}) = F_{X, Y \otimes Z}^{(2)} \circ (\text{id}_{F(X)} \otimes F_{Y,Z}^{(2)}),$$

$$(2.3) \quad F_{X, \mathbf{1}}^{(2)} \circ (\text{id}_{F(X)} \otimes F^{(0)}) = \text{id}_{F(X)} = F_{\mathbf{1}, X}^{(2)} \circ (F^{(0)} \otimes \text{id}_{F(X)})$$

for all objects $X, Y, Z \in \mathcal{C}$. A monoidal functor F is said to be *strong* if $F^{(2)}$ and $F^{(0)}$ are invertible, and *strict* if they are the identities.

2.2. Shibukawa's category of dynamical sets. Let Λ be a non-empty set. To establish a category-theoretical framework dealing with the set-theoretical dynamical Yang-Baxter equation, Shibukawa [19] introduced the category \mathbf{DSet}_Λ . An object of this category is a set X equipped with a map

$$\triangleleft_X : \Lambda \times X \rightarrow X, \quad (\lambda, x) \mapsto \lambda \triangleleft_X x.$$

We will write $\triangleleft = \triangleleft_X$ if no confusion arises. Given two objects X and Y of \mathbf{DSet}_Λ , a morphism $f : X \rightarrow Y$ in \mathbf{DSet}_Λ is a map $f : \Lambda \times X \rightarrow Y$ satisfying

$$(2.4) \quad \lambda \triangleleft f(\lambda, x) = \lambda \triangleleft x$$

for all $\lambda \in \Lambda$, $x \in X$ and $y \in Y$. The composition $g \diamond f$ of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ in \mathbf{DSet}_Λ is defined by

$$(2.5) \quad (g \diamond f)(\lambda, x) = g(\lambda, f(\lambda, x))$$

for $x \in X$ and $\lambda \in \Lambda$. We may regard a morphism $f : X \rightarrow Y$ in \mathbf{DSet}_Λ as a family $\{f(\lambda)\}_{\lambda \in \Lambda}$ of maps from X to Y by $f(\lambda)(x) = f(\lambda, x)$. Then the composition of morphisms in \mathbf{DSet}_Λ is expressed also as

$$(g \diamond f)(\lambda) = g(\lambda) \circ f(\lambda) \quad (\lambda \in \Lambda),$$

where \circ is the usual composition of maps. It is easy to see that the second projection $\Lambda \times X \rightarrow X$ is the identity morphism on the object X . Following Rump's terminology [17],

Definition 2.1. We call \mathbf{DSet}_Λ the category of *dynamical sets* over Λ .

One should be careful when working in \mathbf{DSet}_Λ since a morphism in this category is not a map between the underlying sets. To be sure, we clarify what an isomorphism in this category is:

Lemma 2.2. *A morphism $f : X \rightarrow Y$ in \mathbf{DSet}_Λ is an isomorphism if and only if the map $f(\lambda) : X \rightarrow Y$ is bijective for all $\lambda \in \Lambda$.*

Proof. If f is an isomorphism, then there is a morphism $g : Y \rightarrow X$ in \mathbf{DSet}_Λ such that $f \diamond g$ and $g \diamond f$ are the identity morphisms. By the definition (2.5) of the composition in \mathbf{DSet}_Λ , the map $g(\lambda)$ must be an inverse of $f(\lambda)$. Thus, in particular, the map $f(\lambda)$ is a bijection for all $\lambda \in \Lambda$.

Suppose, conversely, that the map $f(\lambda) : X \rightarrow Y$ is a bijection for all $\lambda \in \Lambda$. We define $g(\lambda) : Y \rightarrow X$ to be the inverse of $f(\lambda)$. Then $g = \{g(\lambda)\}$ is a morphism from Y to X in \mathbf{DSet}_Λ . Indeed, since f is a morphism in \mathbf{DSet}_Λ , we have

$$\lambda \triangleleft g(\lambda, y) = \lambda \triangleleft f(\lambda, g(\lambda, y)) = \lambda \triangleleft (f(\lambda)g(\lambda)(y)) = \lambda \triangleleft y$$

for all $\lambda \in \Lambda$ and $y \in Y$. It is easy to check that f is an isomorphism in \mathbf{DSet}_Λ with inverse g . \square

We define the object $K_\Lambda \in \mathbf{DSet}_\Lambda$ as follows: As a set, $K_\Lambda = \Lambda$. The structure map $\triangleleft_{K_\Lambda}$ for K_Λ is defined by $\lambda \triangleleft_{K_\Lambda} k = k$ for $\lambda \in \Lambda$ and $k \in K$. This object has the following universal property:

Lemma 2.3. $K_\Lambda \in \mathbf{DSet}_\Lambda$ is a terminal object.

Proof. Let $X \in \mathbf{DSet}_\Lambda$ be an object. There is a morphism

$$(2.6) \quad \phi_X : X \rightarrow K_\Lambda, \quad \psi_X(\lambda, x) = \lambda \triangleleft_X x \quad (\lambda \in \Lambda, x \in X).$$

If $f : X \rightarrow K_\Lambda$ be a morphism in \mathbf{DSet}_Λ , then, by (2.4), we have

$$f(\lambda, x) = \lambda \triangleleft_{K_\Lambda} f(\lambda, x) = \lambda \triangleleft_X x = \psi_X(\lambda, x)$$

for all $\lambda \in \Lambda$ and $x \in X$. Thus the set $\mathbf{DSet}_\Lambda(X, K_\Lambda)$ of morphisms from X to K_Λ consists of one element ψ_X defined in the above. \square

Given two objects $X, Y \in \mathbf{DSet}_\Lambda$, we define their tensor product $X \otimes Y \in \mathbf{DSet}_\Lambda$ as the Cartesian product $X \times Y$ endowed with the structure map given by

$$(2.7) \quad \lambda \triangleleft_{X \otimes Y} (x, y) = (\lambda \triangleleft x) \triangleleft y$$

for $\lambda \in \Lambda$, $x \in X$ and $y \in Y$. For morphisms $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ in \mathbf{DSet}_Λ , their tensor product $f \otimes g : X \otimes Y \rightarrow X' \otimes Y'$ is given by

$$(2.8) \quad (f \otimes g)(\lambda, (x, y)) = (f(\lambda, x), g(\lambda \triangleleft x, y))$$

for $\lambda \in \Lambda$, $x \in X$ and $y \in Y$. This definition can be expressed also as

$$(2.9) \quad (f \otimes g)(\lambda) = f(\lambda) \times g(\lambda \triangleleft X^{(1)})$$

if we use the notation similar to that used in the set-theoretical dynamical Yang-Baxter equation (1.4). The singleton $\mathbf{1} := \{1\}$ is an object of \mathbf{DSet}_Λ with the structure map defined by $\lambda \triangleleft 1 = \lambda$ for $\lambda \in \Lambda$. There are natural isomorphisms

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z), \quad \ell_X : \mathbf{1} \otimes X \rightarrow X, \quad r_X : X \otimes \mathbf{1} \rightarrow X$$

for objects $X, Y, Z \in \mathbf{DSet}_\Lambda$ defined by

$$a_{X,Y,Z}(\lambda)((x, y), z) = (x, (y, z)), \quad \ell_X(\lambda)(1, x) = x, \quad r_X(\lambda)(x, 1) = x,$$

respectively, for $\lambda \in \Lambda$, $x \in X$, $y \in Y$ and $z \in Z$. The category \mathbf{DSet}_Λ is a monoidal category with the tensor product \otimes , the unit object $\mathbf{1}$, and the natural isomorphisms defined in the above.

Remark 2.4. The category \mathbf{DSet}_Λ has an alternative tensor product $\check{\otimes}$ defined as follows: For objects $X, Y \in \mathbf{DSet}_\Lambda$, we define $X \check{\otimes} Y \in \mathbf{DSet}_\Lambda$ to be the Cartesian product $X \times Y \in \mathbf{DSet}_\Lambda$ endowed with the structure morphism defined by

$$\lambda \triangleleft (x, y) = (\lambda \triangleleft y) \triangleleft x \quad (\lambda \in \Lambda, x \in X, y \in Y).$$

For morphisms $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ in \mathbf{DSet}_Λ , we set

$$(f \check{\otimes} g)(\lambda, x, y) = f(\lambda \triangleleft y, x) \times g(\lambda, y) \quad (\lambda \in \Lambda, x \in X, y \in Y).$$

There is a natural isomorphism

$$\tau_{X,Y} : X \otimes Y \rightarrow Y \check{\otimes} X, \quad (x, y) \mapsto (y, x) \quad (X, Y \in \mathbf{DSet}_\Lambda).$$

The identity functor on \mathbf{DSet}_Λ gives an isomorphism

$$(\text{id}_{\mathbf{DSet}_\Lambda}, \tau, \text{id}_1) : (\mathbf{DSet}_\Lambda, \otimes^{\text{rev}}, \mathbf{1}) \xrightarrow{\sim} (\mathbf{DSet}_\Lambda, \check{\otimes}, \mathbf{1})$$

of monoidal categories, where $X \otimes^{\text{rev}} Y = Y \otimes X$. The two tensor products \otimes and $\check{\otimes}$ are equivalent in this sense. We use the former one unless otherwise stated.

2.3. Category of quivers. Let Λ be a non-empty set. A *quiver* over Λ is a set Q endowed with two maps $\mathfrak{s}_Q, \mathfrak{t}_Q : Q \rightarrow \Lambda$, called the *source map* and the *target map*, respectively. We write $\mathfrak{s} = \mathfrak{s}_Q$ and $\mathfrak{t} = \mathfrak{t}_Q$ if no confusion arises. An element $a \in Q$ of a quiver Q is called an *arrow* from $\mathfrak{s}(a)$ to $\mathfrak{t}(a)$. The diagram

$$\begin{array}{ccc} \lambda & \xrightarrow{a} & \mu \\ \circ & & \circ \end{array}$$

will be used to mean that a is an arrow from λ to μ .

The category \mathbf{Quiv}_Λ of quivers over Λ is defined as follows: As its name suggests, an object of this category is a quiver over Λ . If Q and Q' are objects of \mathbf{Quiv}_Λ , then a morphism $f : Q \rightarrow Q'$ in \mathbf{Quiv}_Λ is a map $f : Q \rightarrow Q'$ such that

$$\mathfrak{s}(f(a)) = \mathfrak{s}(a) \quad \text{and} \quad \mathfrak{t}(f(a)) = \mathfrak{t}(a)$$

for all elements $a \in Q$. The composition of morphisms in \mathbf{Quiv}_Λ is defined by the ordinary composition of maps. The following lemma is easy to prove:

Lemma 2.5. *A morphism in \mathbf{Quiv}_Λ is an isomorphism in \mathbf{Quiv}_Λ if and only if it is a bijection between the underlying sets.*

Given two objects Q and R of \mathbf{Quiv}_Λ , we define their fiber product $Q \times_\Lambda R$ by

$$Q \times_\Lambda R = \{(a, b) \in Q \times R \mid \mathfrak{t}(a) = \mathfrak{s}(b)\}.$$

The set $Q \times_\Lambda R$ is a quiver over Λ by

$$\mathfrak{s}(a, b) = \mathfrak{s}(a) \quad \text{and} \quad \mathfrak{t}(a, b) = \mathfrak{t}(b) \quad ((a, b) \in Q \times_\Lambda R).$$

Given two morphisms $f : Q \rightarrow Q'$ and $g : R \rightarrow R'$ in \mathbf{Quiv}_Λ , we define the map

$$f \times_\Lambda g : Q \times_\Lambda R \rightarrow Q' \times_\Lambda R'$$

to be the restriction of $f \times g$ to $Q \times_\Lambda R$. The fiber product \times_Λ makes \mathbf{Quiv}_Λ a monoidal category. We note that the unit object of \mathbf{Quiv}_Λ is the set Λ with the source and the target map given by $\mathfrak{s} = \text{id}_\Lambda = \mathfrak{t}$.

Remark 2.6. \mathbf{Quiv}_Λ has an alternative tensor product $\check{\times}_\Lambda$ given by

$$Q \check{\times}_\Lambda R = \{(a, b) \in Q \times R \mid \mathfrak{s}(a) = \mathfrak{t}(b)\}.$$

Given $Q \in \mathbf{Quiv}_\Lambda$, we define the *opposite quiver* \overline{Q} of Q to be the quiver obtained by ‘reversing’ all arrows of Q . Formally, \overline{Q} is same as Q as a set. The source and the target maps of \overline{Q} are defined by $\mathfrak{s}_{\overline{Q}}(\overline{a}) = \mathfrak{t}_Q(a)$ and $\mathfrak{t}_{\overline{Q}}(\overline{a}) = \mathfrak{s}_Q(a)$ for $a \in Q$, respectively, where an element $a \in Q$ is written as \overline{a} when it is regarded as an element of \overline{Q} . The assignment $Q \mapsto \overline{Q}$ gives rise to an isomorphism

$$(\overline{}, \text{id}, \text{id}_\Lambda) : (\mathbf{Quiv}_\Lambda, \times_\Lambda, \Lambda) \xrightarrow{\approx} (\mathbf{Quiv}_\Lambda, \check{\times}_\Lambda, \Lambda)$$

of monoidal categories. The two tensor products \times_Λ and $\check{\times}_\Lambda$ are equivalent in this sense. We always consider the former one in this paper.

2.4. Embedding \mathbf{DSet}_Λ into \mathbf{Quiv}_Λ . Let Λ be a non-empty set. Given an object $X \in \mathbf{DSet}_\Lambda$, we define the quiver $\mathbf{Q}(X)$ over Λ as follows: As a set,

$$\mathbf{Q}(X) = \Lambda \times X.$$

The source and the target map of this quiver are defined by

$$\mathfrak{s}_{\mathbf{Q}(X)}(\lambda, x) = \lambda \quad \text{and} \quad \mathfrak{t}_{\mathbf{Q}(X)}(\lambda, x) = \lambda \triangleleft_X x,$$

respectively, for $\lambda \in \Lambda$ and $x \in X$. Recall that a morphism $f : X \rightarrow Y$ in \mathbf{DSet}_Λ is a map $f : \Lambda \times X \rightarrow Y$ satisfying (2.4). For such an f , we define

$$\mathbf{Q}(f) : \mathbf{Q}(X) \rightarrow \mathbf{Q}(Y), \quad \mathbf{Q}(f)(\lambda, x) = (\lambda, f(\lambda, x)) \quad (\lambda \in \Lambda, x \in X).$$

We thus obtain a functor $\mathbf{Q} : \mathbf{DSet}_\Lambda \rightarrow \mathbf{Quiv}_\Lambda$. Our main result in this section states that the functor \mathbf{Q} embeds the monoidal category \mathbf{DSet}_Λ into \mathbf{Quiv}_Λ . Namely, we prove:

Theorem 2.7. *The functor $\mathbf{Q} : \mathbf{DSet}_\Lambda \rightarrow \mathbf{Quiv}_\Lambda$ is a fully faithful strong monoidal functor together with the monoidal structure given by*

$$(2.10) \quad \mathbf{Q}_{X,Y}^{(2)} : \mathbf{Q}(X) \times_\Lambda \mathbf{Q}(Y) \rightarrow \mathbf{Q}(X \otimes Y), \quad (\lambda, x, \mu, y) \mapsto (\lambda, x, y),$$

$$(2.11) \quad \mathbf{Q}^{(0)} : \mathbf{Q}(\mathbf{1}) \rightarrow \Lambda, \quad (\lambda, 1) \mapsto \lambda.$$

An object $Q \in \mathbf{Quiv}_\Lambda$ is isomorphic to $\mathbf{Q}(X)$ for some object $X \in \mathbf{DSet}_\Lambda$ if and only if the following condition is satisfied:

$$(2.12) \quad \text{The cardinality of the set } \mathfrak{s}_Q^{-1}(\lambda) \text{ does not depend on } \lambda \in \Lambda.$$

Proof. (1) *Fully faithfulness.* Let $f : X \rightarrow Y$ be a morphism in \mathbf{DSet}_Λ . By composing the second projection to $\mathbf{Q}(f)$, we recover the map f itself. Thus \mathbf{Q} is faithful. To show that \mathbf{Q} is full, we let $X, Y \in \mathbf{DSet}_\Lambda$. Given a morphism $\xi : \mathbf{Q}(X) \rightarrow \mathbf{Q}(Y)$ in \mathbf{Quiv}_Λ , we define $f : \Lambda \times X \rightarrow Y$ to be the composition

$$f = \left(\Lambda \times X \xrightarrow{\xi} \Lambda \times Y \xrightarrow{\text{pr}_2} Y \right),$$

where pr_2 is the second projection. Since ξ preserves the source map, we have

$$\xi(\lambda, x) = (\lambda, f(\lambda, x))$$

for all $\lambda \in \Lambda$ and $x \in X$. Since ξ also preserves the target map, we have

$$\lambda \triangleleft f(\lambda, x) = \mathfrak{s}(\xi(\lambda, x)) = \mathfrak{s}(\lambda, x) = \lambda \triangleleft x$$

for all $\lambda \in \Lambda$ and $x \in X$. Namely, f is a morphism from X to Y in \mathbf{DSet}_Λ such that $\xi = \mathbf{Q}(f)$. Hence the functor \mathbf{Q} is full.

(2) *Monoidal structure.* We prove that \mathbf{Q} a strong monoidal functor by (2.10) and (2.11). It is easy to see that $\mathbf{Q}^{(0)}$ is an isomorphism in \mathbf{Quiv}_Λ . Let X and Y be objects of \mathbf{DSet}_Λ . To investigate the properties of $\mathbf{Q}^{(2)}$, we note that

$$\mathbf{Q}(X) \times_\Lambda \mathbf{Q}(Y) = \{(\lambda, x, \mu, y) \in \Lambda \times X \times \Lambda \times Y \mid \mu = \lambda \triangleleft x\}$$

as sets. By this description and Lemma 2.5, it is easy to check that $\mathbf{Q}_{X,Y}^{(2)}$ is an isomorphism in \mathbf{Quiv}_Λ . For any morphisms $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ in \mathbf{DSet}_Λ

and an element $(\lambda, x, \mu, y) \in Q(X) \times_{\Lambda} Q(Y)$, we have

$$\begin{aligned} Q_{X',Y'}^{(2)}(Q(f) \times_{\Lambda} Q(g))(\lambda, x, \mu, y) &= Q_{X',Y'}^{(2)}(\lambda, f(\lambda, x), \mu, g(\mu, y)) \\ &= (\lambda, f(\lambda, x), g(\mu, y)) \\ &= (\lambda, f(\lambda, x), g(\lambda \triangleleft x, y)) \\ &= (\lambda, (f \otimes g)(\lambda, x, y)) \\ &= Q(f \otimes g)Q_{X,Y}^{(2)}(\lambda, x, \mu, y). \end{aligned}$$

Thus $Q_{X,Y}^{(2)}$ is an isomorphism in \mathbf{Quiv}_{Λ} natural in the variables X and Y . It is easy to check that $Q^{(2)}$ and $Q^{(0)}$ satisfy the conditions (2.2) and (2.3).

(3) *Determination of the essential image.* Let \mathcal{I} be the essential image of Q , that is, the full subcategory of \mathbf{Quiv}_{Λ} consisting of all objects $Q \in \mathbf{Quiv}_{\Lambda}$ such that $Q \cong Q(X)$ for some $X \in \mathbf{DSet}_{\Lambda}$. It is easy to see that every object of \mathcal{I} satisfies the condition (2.12). We now suppose that an object $Q \in \mathbf{Quiv}_{\Lambda}$ satisfies (2.12). We fix $\lambda_0 \in \Lambda$ and set $X = \mathfrak{s}_Q^{-1}(\lambda_0)$. For each $\lambda \in \Lambda$, there is a bijection $\phi_{\lambda} : X \rightarrow \mathfrak{s}_Q^{-1}(\lambda)$ by the assumption. Now we view the set X as an object of \mathbf{DSet}_{Λ} by defining

$$\lambda \triangleleft_X x = \mathfrak{t}_Q(\phi_{\lambda}(x)) \quad (\lambda \in \Lambda, x \in X).$$

Then the map $Q(X) \rightarrow Q$ given by $(\lambda, x) \mapsto \phi_{\lambda}(x)$ is an isomorphism in \mathbf{Quiv}_{Λ} . Thus Q belongs to the essential image of Q . \square

2.5. Dynamical Yang-Baxter maps. For a monoidal category \mathcal{C} , the category $\mathbf{Br}(\mathcal{C})$ of *braided objects* of \mathcal{C} is defined as follows: An object of this category is a pair (V, c) consisting of an object $V \in \mathcal{C}$ and a morphism $c : V \otimes V \rightarrow V \otimes V$ in \mathcal{C} satisfying the braid relation

$$(c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes c) \circ (c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c) \circ (c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes c).$$

If (V, c) and (W, d) are objects of $\mathbf{Br}(\mathcal{C})$, then a morphism $f : (V, c) \rightarrow (W, d)$ in $\mathbf{Br}(\mathcal{C})$ is a morphism $f : V \rightarrow W$ in \mathcal{C} such that

$$(f \otimes f) \circ c = d \circ (f \otimes f).$$

Remark 2.8. In this paper, we do not assume that the morphism-part c of a braided object (V, c) is an isomorphism in the underlying category. Thus our notion of a braided object is what is called a *pre-braided object* in [10, 11, 12]. As pointed out in these papers, non-invertible braided objects naturally arise in studying several algebraic structures.

The equation (1.1) says that the pair (V, σ) is a braided object of the category of vector spaces. Similarly, the set-theoretical Yang-Baxter equation (1.2) is equivalent to that the pair (X, σ) is a braided object of the category of sets. The dynamical Yang-Baxter equation (1.3) can also be regarded as a braided object in the monoidal category $\mathcal{V}_{\mathfrak{h}}$ introduced in [5].

The category \mathbf{DSet}_{Λ} has a role of $\mathcal{V}_{\mathfrak{h}}$ in the study of the set-theoretical Yang-Baxter equation (1.4). As we have introduced in Definition 1.1, a *dynamical Yang-Baxter map* on a dynamical set X is a family $\sigma = \{\sigma(\lambda)\}_{\lambda \in \Lambda}$ of maps satisfying (1.4) and (1.5). Shibukawa [19] pointed out that σ is a dynamical Yang-Baxter map on X if and only if (X, σ) is a braided object of \mathbf{DSet}_{Λ} .

Now let Λ be a non-empty set. Since the assignment $\mathcal{C} \mapsto \mathbf{Br}(\mathcal{C})$ is functorial with respect to strong monoidal functors, we have:

Theorem 2.9. *The functor Q of Theorem 2.7 induces a fully faithful functor*

$$\mathrm{Br}(Q) : \mathrm{Br}(\mathrm{DSet}_\Lambda) \rightarrow \mathrm{Br}(\mathrm{Quiv}_\Lambda).$$

In more detail, the functor $\mathrm{Br}(Q)$ sends a braided object (X, σ) of DSet_Λ to the braided object $(Q(X), \tilde{\sigma})$ of Quiv_Λ , where

$$(2.13) \quad \tilde{\sigma} := (Q_{X,X}^{(2)})^{-1} \circ Q(\sigma) \circ Q_{X,X}^{(2)}.$$

If we use the notations given by (1.6), then the morphism $\tilde{\sigma}$ is given by

$$(2.14) \quad \tilde{\sigma}((\lambda, x), (\mu, y)) = ((\lambda, x \xrightarrow{\lambda} y), (\lambda \triangleleft (x \xrightarrow{\lambda} y), x \xleftarrow{\lambda} y))$$

for $\lambda, \mu \in \Lambda$ and $x, y \in X$ with $\mu = \lambda \triangleleft x$.

A braided object of Quiv_Λ is called a *braided quiver* over Λ [1]. The above theorem says that finding a dynamical Yang-Baxter map on $X \in \mathrm{DSet}_\Lambda$ is equivalent to finding a braided quiver (A, σ) such that A is isomorphic to $Q(X)$. In practice, the category Quiv_Λ is easier to deal with than DSet_Λ .

Remark 2.10. Let X be an object of DSet_Λ , and let $\{\tilde{\sigma}(\lambda) : X \times X \rightarrow X \times X\}_{\lambda \in \Lambda}$ be a family of maps. We consider the condition

$$(2.15) \quad (\lambda \triangleleft R_{xy}^{(\lambda)}) \triangleleft L_{xy}^{(\lambda)} = (\lambda \triangleleft y) \triangleleft x \quad (\lambda \in \Lambda, x, y \in X),$$

where $(L_{xy}^{(\lambda)}, R_{xy}^{(\lambda)}) = \tilde{\sigma}(\lambda)(x, y)$. In [22], the equation

$$(2.16) \quad \tilde{\sigma}(\lambda)_{23} \circ \tilde{\sigma}(\lambda \triangleleft X^{(3)})_{12} \circ \tilde{\sigma}(\lambda)_{23} = \tilde{\sigma}(\lambda \triangleleft X^{(3)})_{12} \circ \tilde{\sigma}(\lambda)_{23} \circ \tilde{\sigma}(\lambda \triangleleft X^{(3)})_{12}$$

is called the set-theoretical dynamical Yang-Baxter equation instead of (1.4). Recall from Remark 2.4 that DSet_Λ has the alternative tensor product $\tilde{\otimes}$. The condition (2.15) says that $\tilde{\sigma}$ is a morphism $\tilde{\sigma} : X \tilde{\otimes} X \rightarrow X \tilde{\otimes} X$ in DSet_Λ , and the equation (2.16) is equivalent to

$$(\mathrm{id}_X \tilde{\otimes} \tilde{\sigma})(\tilde{\sigma} \tilde{\otimes} \mathrm{id}_X)(\mathrm{id}_X \tilde{\otimes} \tilde{\sigma}) = (\tilde{\sigma} \tilde{\otimes} \mathrm{id}_X)(\mathrm{id}_X \tilde{\otimes} \tilde{\sigma})(\tilde{\sigma} \tilde{\otimes} \mathrm{id}_X)$$

in $(\mathrm{DSet}_\Lambda, \tilde{\otimes}, \mathbf{1})$. Thus $\tilde{\sigma}$ satisfies (2.15) and (2.16) if and only if

$$\sigma(\lambda)(x, y) = (R_{yx}^{(\lambda)}, L_{yx}^{(\lambda)}) \quad (\lambda \in \Lambda, x, y \in X)$$

is a dynamical Yang-Baxter map on X .

3. SOLUTIONS ARISING FROM LEFT QUASIGROUPS

3.1. Dynamical sets of PH type. We fix a non-empty set Λ throughout this section. A *left quasigroup* is a set L endowed with a binary operation $*_L$ such that the map $L \rightarrow L$, $x \mapsto a *_L x$ is bijective for all $a \in L$. For a left quasigroup L , the *left division* of L is the binary operation \backslash_L on L determined by

$$a \backslash_L (a *_L x) = x = a *_L (a \backslash_L b) \quad (\forall a, x \in L).$$

Many known examples of dynamical Yang-Baxter maps are constructed by using left quasigroups [9, 15, 19, 20, 21]. Motivated by these results, we consider the following class of dynamical sets:

Definition 3.1. We say that a dynamical set $X \in \mathrm{DSet}_\Lambda$ is of *principal homogeneous type*, or *PH type* for short, if the following map is bijective:

$$(3.1) \quad \Lambda \times X \rightarrow \Lambda \times \Lambda, \quad (\lambda, x) \mapsto (\lambda, \lambda \triangleleft x).$$

If $*$ is a binary operation on Λ such that $(\Lambda, *)$ is a left quasigroup, then the set $X = \Lambda$ is a dynamical set over Λ of PH type by $\triangleleft_X = *$. For example, the binary operation $*$ given by $\lambda * \mu = \mu$ makes Λ a left quasigroup, and the resulting dynamical set is the terminal object $K_\Lambda \in \mathbf{DSet}_\Lambda$ mentioned in Lemma 2.3. Every dynamical set over Λ of PH type can be obtained from a left quasigroup structure on Λ in this way. In fact, the following stronger statement holds true:

Lemma 3.2. *An object $X \in \mathbf{DSet}_\Lambda$ is of PH type if and only if $X \cong K_\Lambda$.*

Proof. There is a morphism $\psi_X : X \rightarrow K_\Lambda$ in \mathbf{DSet}_Λ given by (2.6). The map (3.1) is just the morphism $Q(\psi_X) : Q(X) \rightarrow Q(K_\Lambda)$ in \mathbf{Quiv}_Λ . By the faithfulness of Q , it is clear that X is of PH type if and only if ψ_X is an isomorphism. \square

Thus, strange as it may sound, a dynamical set over Λ of PH type is unique up to isomorphism. Nevertheless we have introduced this terminology, since we will deal with many such objects having various different appearances.

3.2. Classification of the solutions. Let X be a dynamical set over Λ of PH type. As an analogue of the left division of a left quasigroup, for elements $\lambda, \lambda' \in \Lambda$, we define the *left division* $\lambda \backslash_X \lambda' \in X$ to be the unique element of X such that

$$(3.2) \quad \lambda \triangleleft (\lambda \backslash_X \lambda') = \lambda'.$$

We recall Shibukawa's classification result of the dynamical Yang-Baxter maps on X in a slightly reformulated form. A *ternary operation* on a set M is just a map from $M \times M \times M$ to M . Suppose that a ternary operation

$$(3.3) \quad \langle -, -, - \rangle : \Lambda \times \Lambda \times \Lambda \rightarrow \Lambda, \quad (a, b, c) \mapsto \langle a, b, c \rangle \quad (a, b, c \in \Lambda)$$

on the set Λ is given. For $\lambda \in \Lambda$ and $x, y \in X$, we define

$$(3.4) \quad \sigma(\lambda)(x, y) = (x \xrightarrow{\lambda} y, x \xleftarrow{\lambda} y),$$

where

$$(3.5) \quad x \xrightarrow{\lambda} y = \lambda \backslash_X \langle \lambda, \lambda \triangleleft x, (\lambda \triangleleft x) \triangleleft y \rangle,$$

$$(3.6) \quad x \xleftarrow{\lambda} y = \langle \lambda, \lambda \triangleleft x, (\lambda \triangleleft x) \triangleleft y \rangle \backslash_X ((\lambda \triangleleft x) \triangleleft y).$$

We remark that the symbols $x \xrightarrow{\lambda} y$ and $x \xleftarrow{\lambda} y$ are defined so that the family

$$(3.7) \quad \sigma := \{\sigma(\lambda) : X \times X \rightarrow X \times X\}_{\lambda \in \Lambda}$$

of maps is a morphism $X \otimes X \rightarrow X \otimes X$ in \mathbf{DSet}_Λ . Shibukawa [21, Theorem 3.2] proved:

Theorem 3.3. *The morphism $\sigma : X \otimes X \rightarrow X \otimes X$ is a dynamical Yang-Baxter map on X if and only if the ternary operation (3.3) satisfies the equations*

$$(3.8) \quad \langle a, \langle a, b, c \rangle, \langle \langle a, b, c \rangle, c, d \rangle \rangle = \langle a, b, \langle b, c, d \rangle \rangle,$$

$$(3.9) \quad \langle \langle a, b, c \rangle, c, d \rangle = \langle \langle a, b, \langle b, c, d \rangle \rangle, \langle b, c, d \rangle, d \rangle$$

for all $a, b, c, d \in \Lambda$. Moreover, every dynamical Yang-Baxter map on X can be obtained in this way from such a ternary operation on Λ .

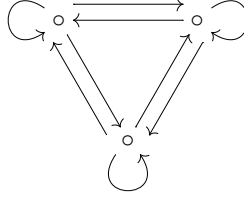


FIGURE 1. Complete quiver over a set of three elements

To avoid confusion with the composition of maps, we use the symbol \diamond to express the composition of morphisms in \mathbf{DSet}_Λ . We say that two dynamical Yang-Baxter maps (X, σ) and (Y, τ) are *equivalent* if they are isomorphic in $\mathbf{Br}(\mathbf{DSet}_\Lambda)$, that is, if there is an isomorphism $f : X \rightarrow Y$ in \mathbf{DSet}_Λ such that

$$(f \otimes f) \diamond \sigma = \tau \diamond (f \otimes f).$$

As in Lemma 2.3, we denote by $\psi_X : X \rightarrow K_\Lambda$ the unique morphism in \mathbf{DSet}_Λ . In this section, we also give the following classification up to equivalence.

Theorem 3.4. *Let X_i ($i = 1, 2$) be a dynamical set over Λ of PH type, and let T_i be a ternary operation on Λ satisfying (3.8) and (3.9). We define σ_i by (3.4)–(3.7) with $X = X_i$ and $\langle -, -, - \rangle = T_i$. Then (X_1, σ_1) and (X_2, σ_2) are equivalent if and only if $T_1 = T_2$. If this is the case, then we have*

$$\sigma_2 = (\psi_{21} \otimes \psi_{21}) \diamond \sigma_1 \diamond (\psi_{12} \otimes \psi_{12}),$$

where $\psi_{ij} = (\psi_{X_i})^{-1} \diamond \psi_{X_j}$ ($i, j = 1, 2$).

Explicitly, the morphism $\psi_{ij} : X_j \rightarrow X_i$ in the above is given by

$$\psi_{ij}(\lambda, x) = \lambda \setminus_{X_i} (\lambda \triangleleft_{X_j} x) \quad (\lambda \in \Lambda, x \in X_j).$$

We give proofs of these theorems by emphasizing the relation between dynamical sets over Λ and quivers over Λ . We will see that the quivers corresponding to the dynamical sets of PH type are of very simple form, and the equations (3.8) and (3.9) naturally arise from the braid equation on such quivers. Our point of view also clarifies relations between the properties of the ternary operation on Λ and the properties of the resulting dynamical Yang-Baxter maps.

3.3. Morphisms on a complete quiver. A *complete quiver* over Λ is a quiver A over Λ such that there exists a unique arrow of A from λ to μ for each pair (λ, μ) of elements of Λ . We warn that a complete quiver is *not* one whose underlying graph is a complete graph. For example, if Λ consists of three elements, a complete quiver over Λ is depicted as in Figure 1.

A complete quiver over Λ is unique up to isomorphism. More precisely, if A is a complete quiver over Λ , then there is an isomorphism

$$A \rightarrow \mathbf{Q}(K), \quad a \mapsto (\mathfrak{s}(a), \mathfrak{t}(a)) \quad (a \in A)$$

of quivers, where $K = K_\Lambda$ is the object of \mathbf{DSet}_Λ discussed in Lemma 2.3. In this sense, a complete quiver can be thought of as a quiver-theoretical counterpart of the notion of a dynamical set of PH type.

Now let A be a complete quiver over Λ . For $\lambda, \mu \in \Lambda$, we denote by $\lambda \rightarrow \mu$ the unique element $a \in A$ such that $\mathfrak{s}(a) = \lambda$ and $\mathfrak{t}(a) = \mu$. Then we have

$$\underbrace{A \times_{\Lambda} \cdots \times_{\Lambda} A}_{m \text{ times}} = \{(\lambda_1 \rightarrow \lambda_2, \lambda_2 \rightarrow \lambda_3, \dots, \lambda_m \rightarrow \lambda_{m+1}) \mid \lambda_1, \dots, \lambda_{m+1} \in \Lambda\}$$

for a positive integer m . For simplicity, we write

$$(3.10) \quad \lambda_1 \rightarrow \lambda_2 \rightarrow \cdots \rightarrow \lambda_m := (\lambda_1 \rightarrow \lambda_2, \lambda_2 \rightarrow \lambda_3, \dots, \lambda_{m-1} \rightarrow \lambda_m).$$

Let $f : A \times_{\Lambda} A \rightarrow A \times_{\Lambda} A$ be a morphism in \mathbf{Quiv}_{Λ} , and let $\lambda, \mu, \nu \in \Lambda$. Since f preserves the source and the target maps, there is a unique element $\mu' \in \Lambda$ such that $f(\lambda \rightarrow \mu \rightarrow \nu) = \lambda \rightarrow \mu' \rightarrow \nu$. We thus obtain a ternary operation $\langle -, -, - \rangle_f$ on Λ uniquely determined by the property that

$$f(\lambda \rightarrow \mu \rightarrow \nu) = a \rightarrow \langle \lambda, \mu, \nu \rangle_f \rightarrow \nu$$

for all $\lambda, \mu, \nu \in \Lambda$. It is obvious that there holds:

Lemma 3.5. *The assignment $f \mapsto \langle -, -, - \rangle_f$ gives a bijection*

$$(3.11) \quad \mathbf{Quiv}_{\Lambda}(A \times_{\Lambda} A, A \times_{\Lambda} A) \xrightarrow{\cong} \mathbf{Map}(\Lambda \times \Lambda \times \Lambda, \Lambda).$$

3.4. Quiver-theoretical aspect of the construction. Let X be a dynamical set over Λ of PH type. We explain that the formulas (3.4)–(3.7) naturally arise from our quiver-theoretical approach. A key observation is that $A := \mathbf{Q}(X)$ is a complete quiver over Λ . Since the functor \mathbf{Q} is fully faithful, we have a bijection

$$(3.12) \quad \begin{aligned} \mathbf{DSet}_{\Lambda}(X \otimes X, X \otimes X) &\rightarrow \mathbf{Quiv}_{\Lambda}(A \times_{\Lambda} A, A \times_{\Lambda} A), \\ f &\mapsto (\mathbf{Q}_{X,X}^{(2)})^{-1} \circ \mathbf{Q}(f) \circ \mathbf{Q}_{X,X}^{(2)}, \end{aligned}$$

where $\mathbf{Q}^{(2)}$ is the monoidal structure of \mathbf{Q} . We now obtain a bijection

$$(3.13) \quad \mathbf{Map}(\Lambda \times \Lambda \times \Lambda, \Lambda) \xrightarrow{\cong} \mathbf{DSet}_{\Lambda}(X \otimes X, X \otimes X)$$

by composing the bijections (3.11) and (3.12). This bijection clarifies the meaning of the defining formulas (3.4)–(3.7) of the morphism σ of Theorem 3.3.

Lemma 3.6. *The morphism $\sigma : X \otimes X \rightarrow X \otimes X$ in \mathbf{DSet}_{Λ} corresponding to a ternary operation $\langle -, -, - \rangle$ on Λ via (3.13) is given by the formulas (3.4)–(3.7).*

Proof. Let $\langle -, -, - \rangle$ be a ternary operation on Λ , and let $\sigma : X \otimes X \rightarrow X \otimes X$ be the morphism in \mathbf{DSet}_{Λ} corresponding to $\langle -, -, - \rangle$ via (3.13). We write

$$\sigma(\lambda)(x, y) = (L_{xy}^{(\lambda)}, R_{xy}^{(\lambda)})$$

for $\lambda \in \Lambda$ and $x, y \in X$. Let $\tilde{\sigma} : A \times_{\Lambda} A \rightarrow A \times_{\Lambda} A$ be the morphism corresponding to σ via (3.12). Then we have

$$(3.14) \quad \tilde{\sigma}((\lambda, x), (\mu, y)) = ((\lambda, L_{xy}^{(\lambda)}), (\lambda \triangleleft L_{xy}^{(\lambda)}, R_{xy}^{(\lambda)}))$$

for $\lambda, \mu \in \Lambda$ and $x, y \in X$ with $\mu = \lambda \triangleleft x$.

By definition, $\tilde{\sigma}$ corresponds to the ternary operation $\langle -, -, - \rangle$ via (3.11). As in the previous subsection, we use the notation (3.10) to express elements of the power of $A := \mathbf{Q}(X)$ with respect to \times_{Λ} . We then have

$$(3.15) \quad \lambda \rightarrow (\lambda \triangleleft x) = (\lambda, x) \quad \text{and} \quad \lambda \rightarrow \lambda' = (\lambda, \lambda \setminus_X \lambda') \quad (x \in X, \lambda, \lambda' \in \Lambda),$$

where \backslash_X is the left division defined by (3.2). By (3.8), (3.10) and (3.15), we also have the following expression of $\tilde{\sigma}$:

$$\begin{aligned}
 \tilde{\sigma}(a \rightarrow b \rightarrow c) &= a \rightarrow \langle a, b, c \rangle \rightarrow c \\
 (3.16) \quad &= (a \rightarrow \langle a, b, c \rangle, \langle a, b, c \rangle \rightarrow c) \\
 &= ((a, a \backslash_X \langle a, b, c \rangle), (\langle a, b, c \rangle, \langle a, b, c \rangle \backslash_X c)).
 \end{aligned}$$

Now we consider the case where $a = \lambda$, $b = \lambda \triangleleft x$ and $c = (\lambda \triangleleft x) \triangleleft y$ for some $\lambda \in \Lambda$ and $x, y \in X$. Then, by (3.15), we have

$$a \rightarrow b \rightarrow c = ((\lambda, x), (\lambda \triangleleft x, y))$$

as elements of $A \times_\Lambda A$. Comparing (3.14) with (3.16), we conclude that

$$\begin{aligned}
 L_{xy}^{(\lambda)} &= a \backslash_X \langle a, b, c \rangle = \lambda \backslash_X \langle \lambda, \lambda \triangleleft x, (\lambda \triangleleft x) \triangleleft y \rangle \\
 R_{xy}^{(\lambda)} &= \langle a, b, c \rangle \backslash_X c = \langle \lambda, \lambda \triangleleft x, (\lambda \triangleleft x) \triangleleft y \rangle \backslash_X ((\lambda \triangleleft x) \triangleleft y).
 \end{aligned}$$

Thus σ is given by the formulas (3.4)–(3.7). \square

3.5. Proof of Theorem 3.3. Let, as in Theorem 3.3, X be a dynamical set over Λ of PH type. Now we give a proof of Shibukawa's classification result on the dynamical Yang-Baxter maps on X .

We denote by $A = \mathbf{Q}(X)$ the corresponding quiver. Let $\sigma : X \otimes X \rightarrow X \otimes X$ be a morphism in \mathbf{DSet}_Λ , and let $\tilde{\sigma} : A \times_\Lambda A \rightarrow A \times_\Lambda A$ be the morphism corresponding to σ via (3.12). The argument of the previous subsection shows that there is a unique ternary operation $\langle -, -, - \rangle$ on Λ such that the morphism σ is expressed by (3.4)–(3.7). The morphism $\tilde{\sigma}$ is also expressed in terms of the same ternary operation on Λ . The point is that, compared to σ , the morphism $\tilde{\sigma}$ has fairly easier expression: It can be computed by the following rule:

$$(3.17) \quad \tilde{\sigma}(a \rightarrow b \rightarrow c) = a \rightarrow \langle a, b, c \rangle \rightarrow c \quad (a, b, c \in \Lambda).$$

Proof of Theorem 3.3. Since \mathbf{Q} is a strong monoidal functor, the morphism σ is a dynamical Yang-Baxter map if and only if the morphism $\tilde{\sigma}$ satisfies the braid relation. Now we set $\tilde{\sigma}_{12} = \tilde{\sigma} \times_\Lambda \text{id}_A$ and $\tilde{\sigma}_{23} = \text{id}_A \times_\Lambda \tilde{\sigma}$. By the rule (3.17), we compute, without any difficulty,

$$\begin{aligned}
 &\tilde{\sigma}_{12}\tilde{\sigma}_{23}\tilde{\sigma}_{12}(a \rightarrow b \rightarrow c \rightarrow d) \\
 &= \tilde{\sigma}_{12}\tilde{\sigma}_{23}(a \rightarrow \langle a, b, c \rangle \rightarrow c \rightarrow d) \\
 &= \tilde{\sigma}_{12}(a \rightarrow \langle a, b, c \rangle \rightarrow \langle \langle a, b, c \rangle, c, d \rangle \rightarrow d) \\
 &= a \rightarrow \langle a, \langle a, b, c \rangle, \langle \langle a, b, c \rangle, c, d \rangle \rangle \rightarrow \langle \langle a, b, c \rangle, c, d \rangle \rightarrow d, \\
 &\tilde{\sigma}_{23}\tilde{\sigma}_{12}\tilde{\sigma}_{23}(a \rightarrow b \rightarrow c \rightarrow d) \\
 &= \tilde{\sigma}_{23}\tilde{\sigma}_{12}(a \rightarrow b \rightarrow \langle b, c, d \rangle \rightarrow d) \\
 &= \tilde{\sigma}_{23}(a \rightarrow \langle a, b, \langle b, c, d \rangle \rangle \rightarrow \langle b, c, d \rangle \rightarrow d) \\
 &= a \rightarrow \langle a, b, \langle b, c, d \rangle \rangle \rightarrow \langle \langle a, b, \langle b, c, d \rangle \rangle, \langle b, c, d \rangle, d \rangle \rightarrow d
 \end{aligned}$$

for $a, b, c, d \in \Lambda$. Thus σ is a dynamical Yang-Baxter map if and only if (3.8) and (3.9) are satisfied for all $a, b, c, d \in \Lambda$. By Lemma 3.6, every dynamical Yang-Baxter map on X is obtained in this way from a ternary operation on Λ . \square

3.6. Proof of Theorem 3.4. We give a proof of Theorem 3.4. Let $K = K_\Lambda$ be the terminal object considered in Lemma 2.3. For $X = K$, Theorem 3.3 reduces to the following form: Given a ternary operation $\langle -, -, - \rangle$ on Λ , we define

$$(3.18) \quad \sigma_0(\lambda)(x, y) = (\langle \lambda, x, y \rangle, y) \quad (\lambda \in \Lambda, x, y \in K).$$

Then $\sigma_0 = \{\sigma_0(\lambda) : K \times K \rightarrow K \times K\}_{\lambda \in \Lambda}$ is a dynamical Yang-Baxter map on K if and only if the ternary operation $\langle -, -, - \rangle$ satisfies the equations (3.8) and (3.9) of Theorem 3.3. Moreover, every dynamical Yang-Baxter map on K is obtained in this way from such a ternary operation.

Now let X be a dynamical set over Λ of PH type. Since there is an isomorphism $\psi_X : X \rightarrow K$ in \mathbf{DSet}_Λ , the morphism $\sigma_0 : K \otimes K \rightarrow K \otimes K$ induces a morphism $X \otimes X \rightarrow X \otimes X$. We express the induced morphism in an explicit way:

Lemma 3.7. *Let $\langle -, -, - \rangle$ be a ternary operation on Λ , and let σ_0 be the morphism defined by (3.18). Then the morphism*

$$\sigma = (\psi_X^{-1} \otimes \psi_X^{-1}) \diamond \sigma_0 \diamond (\psi_X \otimes \psi_X) : X \otimes X \rightarrow X \otimes X$$

coincides with the morphism defined by the formulas (3.4)–(3.7) by using the same ternary operation on Λ .

This lemma can be proved by the direct computation. To avoid technical computation in \mathbf{DSet}_Λ , and to give a deeper understanding to the background algebraic materials, we rather prove this lemma from the viewpoint of the complete quivers corresponding to X and K .

Proof. Set $A = Q(X)$ and $B = Q(K)$. For $Q = A, B$, we denote by “ $\lambda \rightarrow \mu$ in Q ” the unique arrow of Q starting from λ ending at μ . We extend this notation as in (3.10) and have an obvious isomorphism

$$\Psi : A \times_\Lambda A \rightarrow B \times_\Lambda B, \quad (\lambda \rightarrow \mu \rightarrow \nu \text{ in } A) \mapsto (\lambda \rightarrow \mu \rightarrow \nu \text{ in } B).$$

We define $\tilde{\sigma} : A \times_\Lambda A \rightarrow A \times_\Lambda A$ and $\tilde{\sigma}_0 : B \times_\Lambda B \rightarrow B \times_\Lambda B$ by

$$\tilde{\sigma} = (Q_{X,X}^{(2)})^{-1} \circ Q(\sigma) \circ Q_{X,X}^{(2)} \quad \text{and} \quad \tilde{\sigma}_0 = (Q_{K,K}^{(2)})^{-1} \circ Q(\sigma_0) \circ Q_{K,K}^{(2)},$$

respectively. Now we consider the following diagram:

$$\begin{array}{ccccccc} Q(X \otimes X) & \xrightarrow{(Q_{X,X}^{(2)})^{-1}} & A \times_\Lambda A & \xrightarrow{\Psi} & B \times_\Lambda B & \xrightarrow{Q_{K,K}^{(2)}} & Q(K \otimes K) \\ Q(\sigma) \downarrow & & \tilde{\sigma} \downarrow & & \tilde{\sigma}_0 \downarrow & & \downarrow Q(\sigma_0) \\ Q(X \otimes X) & \xrightarrow{(Q_{X,X}^{(2)})^{-1}} & A \times_\Lambda A & \xrightarrow{\Psi} & B \times_\Lambda B & \xrightarrow{Q_{K,K}^{(2)}} & Q(K \otimes K) \end{array}$$

The left and the right squares commute. We use the rule (3.17) to compute

$$\tilde{\sigma}(\lambda \rightarrow \mu \rightarrow \nu \text{ in } A) = (\lambda \rightarrow \langle \lambda, \mu, \nu \rangle \rightarrow \nu \text{ in } A),$$

$$\tilde{\sigma}_0(\lambda \rightarrow \mu \rightarrow \nu \text{ in } B) = (\lambda \rightarrow \langle \lambda, \mu, \nu \rangle \rightarrow \nu \text{ in } B)$$

for $\lambda, \mu, \nu \in \Lambda$. Hence the middle square of the above diagram also commutes. Now we express Ψ explicitly: For $\lambda, \mu \in \Lambda$ and $x, y \in X$ with $\mu = \lambda \triangleleft x$, we have

$$\begin{aligned} \Psi((\lambda, x), (\mu, y)) &= \Psi(\lambda \rightarrow \mu \rightarrow (\mu \triangleleft y) \text{ in } A) \\ &= (\lambda \rightarrow \mu \rightarrow (\mu \triangleleft y) \text{ in } B) \\ &= ((\lambda, \mu), (\mu, \mu \triangleleft y)) \\ &= ((\lambda, \lambda \triangleleft x), (\mu, \mu \triangleleft y)). \end{aligned}$$

Namely, $\Psi = Q(\psi_X) \times_{\Lambda} Q(\psi_X)$. By the functorial property of $Q^{(2)}$,

$$Q_{K,K}^{(2)} \circ \Psi \circ (Q_{X,X}^{(2)})^{-1} = Q(\psi_X \otimes \psi_X) \circ Q_{X,X}^{(2)} \circ (Q_{X,X}^{(2)})^{-1} = Q(\psi_X \otimes \psi_X).$$

Hence, by the above commutative diagram, we have

$$Q(\sigma) \circ Q(\psi_X \otimes \psi_X) = Q(\psi_X \otimes \psi_X) \circ Q(\sigma_0).$$

Now the claim of this theorem follows from the faithfulness of Q . \square

Proof of Theorem 3.4. The proof boils down to the case where $X_1 = X_2 = K$ by the above lemma. If $X_1 = X_2 = K$, the claim easily follows from the fact that there are no morphisms other than the identity morphism. \square

3.7. Classification of some subclasses of the solutions. Let X be a dynamical set over Λ of PH type. Our method is effective to characterize several classes of dynamical Yang-Baxter maps on X . Let $\langle -, -, - \rangle$ be a ternary operation on Λ satisfying (3.8) and (3.9), and let $\sigma : X \otimes X \rightarrow X \otimes X$ be the dynamical Yang-Baxter map arising from the ternary operation $\langle -, -, - \rangle$.

Theorem 3.8. (a) σ is unitary (i.e., $\sigma \diamond \sigma$ is the identity) if and only if

$$\langle a, \langle a, b, c \rangle, c \rangle = b \quad (\forall a, b, c \in \Lambda).$$

(b) σ is idempotent (i.e., $\sigma \diamond \sigma = \sigma$) if and only if

$$\langle a, \langle a, b, c \rangle, c \rangle = \langle a, b, c \rangle \quad (\forall a, b, c \in \Lambda).$$

(c) σ is invertible if and only if the following map is bijective for all $a, b \in \Lambda$:

$$\Lambda \rightarrow \Lambda, \quad x \mapsto \langle a, x, c \rangle.$$

Part (a) of this theorem is [21, Proposition 7.1].

Proof. Let $\tilde{\sigma} : Q(X) \times_{\Lambda} Q(X) \rightarrow Q(X) \times_{\Lambda} Q(X)$ be the morphism in Quiv_{Λ} corresponding to σ via (3.12). Then the dynamical Yang-Baxter map σ is unitary if and only if $\tilde{\sigma} \circ \tilde{\sigma}$ is the identity. Part (a) follows from

$$\tilde{\sigma}\tilde{\sigma}(a \rightarrow b \rightarrow c) = a \rightarrow \langle a, \langle a, b, c \rangle, c \rangle \rightarrow c \quad (a, b, c \in \Lambda).$$

Part (b) is proved by the same computation. Part (c) follows from the fact that σ is invertible if and only if $\tilde{\sigma}$ is. \square

3.8. Solutions of vertex type. A dynamical Yang-Baxter map (X, σ) is of *vertex type* [21] if the map $\sigma(\lambda)$ does not depend on the parameter $\lambda \in \Lambda$. We note that the class of the dynamical Yang-Baxter maps of vertex type is *not* closed under the equivalence, as the following example shows:

Example 3.9 (Shibukawa [21, Section 8]). Let $*_i$ ($i = 1, 2$) be a binary operation on Λ such that $(\Lambda, *_i)$ is a left quasigroup with left division $/_i$. We define the dynamical set X_i of PH type to be the set $X_i = \Lambda$ equipped with the structure map $*_i : \Lambda \times X_i \rightarrow \Lambda$. Suppose that the first operation $*_1$ satisfies

$$(3.19) \quad (a *_1 c) \backslash_1 ((a *_1 b) *_1 c) = (a' *_1 c) \backslash_1 ((a' *_1 b) *_1 c)$$

for all $a, a', b, c \in \Lambda$ (this condition is satisfied if, for example, Λ is a group with respect to $*_1$). We require nothing of $*_2$. The ternary operation

$$\langle a, b, c \rangle_1 := a *_1 (b \backslash_1 c) \quad (a, b, c \in \Lambda)$$

satisfies (3.8) and (3.9). Now we define $\sigma_i : X_i \otimes X_i \rightarrow X_i \otimes X_i$ ($i = 1, 2$) by the formulas (3.4)–(3.7) with $X = X_i$, $\triangleleft = *_i$ and $\langle -, -, - \rangle = \langle -, -, - \rangle_1$. Theorem 3.3

says that (X_1, σ_1) and (X_2, σ_2) are dynamical Yang-Baxter maps of PH type. The former is given by

$$\sigma_1(\lambda)(x, y) = (y, (\lambda *_1 y) \setminus_1 ((\lambda *_1 x) *_1 y)) \quad (\lambda \in \Lambda, x, y \in X_1),$$

which is a vertex type by virtue of (3.19). The latter is given by

$$\begin{aligned} \sigma_2(\lambda)(x, y) = & (\lambda \setminus_2 (\lambda *_1 ((\lambda *_2 x) \setminus_1 ((\lambda *_2 x) *_2 y))), \\ & (\lambda *_1 ((\lambda *_2 x) \setminus_1 ((\lambda *_2 x) *_2 y))) \setminus_2 ((\lambda *_2 x) *_2 y)) \end{aligned}$$

for $\lambda \in \Lambda$ and $x, y \in X$, which is not a vertex type in general (see [21] for a concrete example).

Following Shibukawa [21], we say that a *dynamical Yang-Baxter map* (X, σ) has a *vertex-IRF correspondence* if it is equivalent to a dynamical Yang-Baxter map of vertex type. It is difficult to determine whether a given dynamical Yang-Baxter map has a vertex-IRF correspondence. For future study of this kind of problems, we give the following new examples of dynamical Yang-Baxter maps having a vertex-IRF correspondence.

Example 3.10. Let \sharp be a binary operation on Λ satisfying

$$(3.20) \quad b \sharp (c \sharp d) = (b \sharp c) \sharp (c \sharp d) \quad \text{and} \quad c \sharp d = (c \sharp d) \sharp d$$

for all $b, c, d \in \Lambda$ (examples of such a binary operation will be given below). Since the ternary operation $\langle a, b, c \rangle = b \sharp c$ ($a, b, c \in \Lambda$) satisfies (3.8) and (3.9),

$$\sigma_0 : K \otimes K \rightarrow K \otimes K, \quad \sigma_0(\lambda)(x, y) = (x \sharp y, y) \quad (\lambda \in \Lambda, x, y \in K)$$

is a dynamical Yang-Baxter map of vertex type on the dynamical set $K = K_\Lambda$ of Lemma 2.3. Now we suppose that we are given a binary operation $*$ on Λ such that $(\Lambda, *)$ is a left quasigroup. Let $X = \Lambda$ be the dynamical set with the structure map given by $\triangleleft_X = *$. We have a dynamical Yang-Baxter map

$$\begin{aligned} \sigma(\lambda)(x, y) = & (\lambda \setminus_X ((\lambda * x) \sharp ((\lambda * x) * y)), \\ & ((\lambda * x) \sharp ((\lambda * x) * y)) \setminus_X ((\lambda * x) * y)) \quad (\lambda \in \Lambda, x, y \in X) \end{aligned}$$

on X , which is equivalent to σ_0 by Theorem 3.4. Unlike Example 3.9, σ is always idempotent by Theorem 3.8.

The problem is to find a binary operation satisfying (3.20). A *band*, introduced by Clifford [2], is a set S endowed with an associative binary operation \vee such that $a \vee a = a$ for all elements $a \in S$. It is easy to verify that the binary operation of a band satisfies (3.20). A familiar example of bands may be a *semilattice*, which is an abstraction of a partially ordered set having a least upper bound of any pair of its elements. Algebraically, a semilattice is just a band (S, \vee) such that $a \vee b = b \vee a$ for all $a, b \in S$.

For example, the set $\Lambda = \mathbb{R}$ of all real numbers is a semilattice by the binary operation \vee defined by $a \vee b = \max\{a, b\}$. We use this semilattice to claim that the above construction produces an idempotent dynamical Yang-Baxter map $\sigma = \{\sigma(\lambda)\}_{\lambda \in \Lambda}$ depending on the parameter $\lambda \in \Lambda$. We make the set $X = \Lambda (= \mathbb{R})$ a left quasigroup by the binary operation

$$x * y := (\text{the middle point of } x \text{ and } y) = \frac{x + y}{2} \quad (x, y \in X).$$

To compute $\sigma(\lambda)$ with $\sharp = \vee$, we shall note

$$(\lambda * x) * y - \lambda * x = \frac{1}{2}y - \frac{1}{4}\lambda - \frac{1}{4}x = \frac{1}{2}(y - \lambda * x).$$

From this equation, we have

$$((\lambda * x) \vee ((\lambda * x) * y)) = \begin{cases} \frac{1}{4}\lambda + \frac{1}{4}x + \frac{1}{2}y & \text{if } y \geq \lambda * x, \\ \frac{1}{2}\lambda + \frac{1}{2}x & \text{otherwise.} \end{cases}$$

The left division of $(X, *)$ is given by $a \backslash b = 2b - a$ for $a, b \in X$. Thus,

$$\sigma(\lambda)(x, y) = \left(-\frac{1}{2}\lambda + \frac{1}{2}x + y, \frac{1}{4}\lambda + \frac{1}{4}x + \frac{1}{2}y \right)$$

for $x, y \in X$ if $y \geq \lambda * x$, and $\sigma(\lambda)(x, y) = (x, y)$ otherwise.

4. A NEW CLASS OF DYNAMICAL YANG-BAXTER MAPS

4.1. A new class of dynamical Yang-Baxter maps. In this section, we give a new class of dynamical Yang-Baxter maps by using the quiver-theoretical techniques developed in the above. Let Λ be a non-empty set. We first prepare the dynamical set $X \in \mathbf{DSet}_\Lambda$ underlying the dynamical Yang-Baxter map that we will construct. We fix a non-empty set X_0 and a family $\{*_i\}_{i \in X_0}$ of binary operations on Λ indexed by X_0 such that $(\Lambda, *_i)$ is a left quasigroup for all $i \in X_0$. We set $X = X_0 \times \Lambda$ and make it a dynamical set over Λ by the structure map defined by

$$\triangleleft_X : \Lambda \times X \rightarrow \Lambda, \quad \lambda \triangleleft (i, \mu) = \lambda *_i \mu \quad (\lambda, \mu \in \Lambda, i \in X_0).$$

Our construction requires the following two data:

- A map $\sigma_0 : X_0 \times X_0 \rightarrow X_0 \times X_0$.
- A ternary operation $\langle -, -, - \rangle$ on Λ .

For $i, j \in X_0$, we write $\sigma_0(i, j) = (i \triangleright j, i \triangleleft j)$. We denote by \backslash_i the left division of the left quasigroup $(\Lambda, *_i)$. We now define $\sigma : X \otimes X \rightarrow X \otimes X$ in \mathbf{DSet}_Λ by

$$(4.1) \quad \sigma(\lambda)((i, x), (j, y)) = \left((i, x) \xrightarrow{\lambda} (j, y), (i, x) \xleftarrow{\lambda} (j, y) \right)$$

for $(i, x), (j, y) \in X = X_0 \times \Lambda$, where

$$(4.2) \quad (i, x) \xrightarrow{\lambda} (j, y) = \left(i \triangleright j, \lambda \backslash_{i \triangleright j} \langle \lambda, \lambda *_i x, (\lambda *_i x) *_j y \rangle \right),$$

$$(4.3) \quad (i, x) \xleftarrow{\lambda} (j, y) = \left(i \triangleleft j, \langle \lambda, \lambda *_i x, (\lambda *_i x) *_j y \rangle \backslash_{i \triangleleft j} ((\lambda *_i x) *_j y) \right).$$

The main theorem of this section is:

Theorem 4.1. *The morphism $\sigma = \{\sigma(\lambda)\}$ is a dynamical Yang-Baxter map on X if and only if σ_0 is a Yang-Baxter map on X_0 and the ternary operation $\langle -, -, - \rangle$ satisfies (3.8) and (3.9) of Theorem 3.3.*

Thus, in a sense, one can obtain a dynamical Yang-Baxter map on X by composing a dynamical Yang-Baxter map on X_0 and a dynamical Yang-Baxter map arising from a ternary operation on Λ .

Theorem 3.3 is the case where X_0 is a singleton. Theorem 3.8 classifies some classes of dynamical Yang-Baxter maps on a dynamical set of PH type in terms of the associated ternary operator. A similar result holds for the dynamical Yang-Baxter maps constructed by the above theorem.

Theorem 4.2. *Suppose that σ_0 is a Yang-Baxter map on X_0 and the ternary operation $\langle -, -, - \rangle$ on Λ satisfies (3.8) and (3.9) of Theorem 3.3 so that the morphism σ defined by (4.1)–(4.3) is a dynamical Yang-Baxter map. Then there holds:*

(a) σ is unitary if and only if σ_0 is unitary and

$$\langle a, \langle a, b, c \rangle, c \rangle = b \quad (\forall a, b, c \in \Lambda).$$

(b) σ is idempotent if and only if σ_0 is idempotent and

$$\langle a, \langle a, b, c \rangle, c \rangle = \langle a, b, c \rangle \quad (\forall a, b, c \in \Lambda).$$

(c) σ is invertible if and only if σ_0 is invertible and the map

$$\Lambda \rightarrow \Lambda, \quad x \mapsto \langle a, x, c \rangle$$

is bijective for all $a, b \in \Lambda$.

4.2. Proof of Theorems 4.1 and 4.2. We keep the notation as above. To prove Theorem 4.1, we consider the quiver $A \in \text{Quiv}_\Lambda$ defined by

$$A = \Lambda \times X_0 \times \Lambda, \quad \mathfrak{s}_A(\lambda, i, \mu) = \lambda \quad \text{and} \quad \mathfrak{t}_A(\lambda, i, \mu) = \mu.$$

We define $K \in \text{DSet}_\Lambda$ to be the set $K = X_0 \times \Lambda$ with the structure map given by $\lambda \triangleleft_K (i, \mu) = \mu$. The quiver A is naturally identified with $\mathbf{Q}(K)$. The dynamical set X introduced in the above is isomorphic to K (cf. Lemma 3.2). More precisely, there holds:

Lemma 4.3. *For $\lambda \in \Lambda$, we define the map $\psi(\lambda) : X \rightarrow K$ by*

$$\psi(\lambda)(i, \lambda') = (i, \lambda *_i \lambda') \quad (\lambda, \lambda' \in \Lambda, i \in X_0).$$

Then $\psi = \{\psi(\lambda)\}_{\lambda \in \Lambda}$ is an isomorphism $\psi : X \rightarrow K$ in DSet_Λ with inverse

$$\psi^{-1} : K \rightarrow X, \quad \psi^{-1}(\lambda)(i, \lambda') = (i, \lambda \setminus_i \lambda') \quad (\lambda, \lambda' \in \Lambda, i \in X_0).$$

This result implies that the dynamical Yang-Baxter maps on X are in one-to-one correspondence to the set of morphisms $\tilde{\sigma} : A \times_\Lambda A \rightarrow A \times_\Lambda A$ in Quiv_Λ such that $(A, \tilde{\sigma})$ is a braided quiver. Now let $\tilde{\sigma} : A \times_\Lambda A \rightarrow A \times_\Lambda A$ be a morphism in Quiv_Λ . For simplicity, we write an arrow $(\lambda, i, \mu) \in A$ as $\lambda \xrightarrow{i} \mu$ (the arrows are “colored” by X_0 unlike the setting of the previous section). We extend this notation as in (3.10). Then, since

$$A \times_\Lambda A = \{\lambda \xrightarrow{i} \mu \xrightarrow{j} \nu \mid \lambda, \mu, \nu \in \Lambda, i, j \in X_0\},$$

one can express the morphism $\tilde{\sigma}$ as

$$(4.4) \quad \tilde{\sigma}(\lambda \xrightarrow{i} \mu \xrightarrow{j} \nu) = \lambda \xrightarrow{t_1(\lambda, i, \mu, j, \nu)} t_2(\lambda, i, \mu, j, \nu) \xrightarrow{t_3(\lambda, i, \mu, j, \nu)} \nu$$

by using a triple $(t_1 : M \rightarrow X_0, t_2 : M \rightarrow \Lambda, t_3 : M \rightarrow X_0)$ of maps from M , where $M = \Lambda \times X_0 \times \Lambda \times X_0 \times \Lambda$. In other words:

Lemma 4.4 (cf. Lemma 3.5). *There is a bijection*

$$\text{Quiv}_\Lambda(A \times_\Lambda A, A \times_\Lambda A) \cong \text{Map}(M, X_0) \times \text{Map}(M, \Lambda) \times \text{Map}(M, X_0).$$

One can therefore, in principle, write down the necessary and sufficient condition for $\tilde{\sigma}$ to satisfy the braid relation in terms of the triple (t_1, t_2, t_3) . The result would be inconvenient and far from understandable. For this reason, we make an ansatz

that the maps t_1 , t_2 and t_3 are expressed by the map $\sigma_0 : X_0 \times X_0 \rightarrow X_0 \times X_0$ and the ternary operation $\langle -, -, - \rangle$ on Λ in the following way:

$$(t_1(\lambda, i, \mu, j, \nu), t_3(\lambda, i, \mu, j, \nu)) = \sigma_0(i, j), \quad t_2(\lambda, i, \mu, j, \nu) = \langle \lambda, \mu, \nu \rangle.$$

Then the formula (4.4) reduces to the following form:

$$(4.5) \quad \tilde{\sigma}(\lambda \xrightarrow{i} \mu \xrightarrow{j} \nu) = \lambda \xrightarrow{i \triangleright j} \langle \lambda, \mu, \nu \rangle \xrightarrow{i \triangleleft j} \nu.$$

The morphisms $\tilde{\sigma}_{12}\tilde{\sigma}_{23}\tilde{\sigma}_{12}$ and $\tilde{\sigma}_{23}\tilde{\sigma}_{12}\tilde{\sigma}_{23}$ can be computed in a similar way as the proof of Theorem 3.3; see Subsection 3.5. As a consequence, we have:

Lemma 4.5. *Suppose that $\tilde{\sigma}$ is given by (4.5). Then $(A, \tilde{\sigma})$ is a braided quiver if and only if σ_0 is a Yang-Baxter map and the ternary operation $\langle -, -, - \rangle$ satisfies (3.8) and (3.9) of Theorem 3.3.*

Now we give a proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\sigma : X \otimes X \rightarrow X \otimes X$ be the morphism in \mathbf{DSet}_Λ defined by (4.1)–(4.3) from the map $\sigma_0 : X_0 \times X_0 \rightarrow X_0 \times X_0$ and the ternary operation $\langle -, -, - \rangle$ on Λ . We consider the diagram

$$\begin{array}{ccccc} Q(X \otimes X) & \xrightarrow{(Q_{X,X}^{(2)})^{-1}} & Q(X) \times_\Lambda Q(X) & \xrightarrow{Q(\psi) \times_\Lambda Q(\psi)} & Q(K) \times_\Lambda Q(K) \\ Q(\sigma) \downarrow & & & & \downarrow \tilde{\sigma} \\ Q(X \otimes X) & \xrightarrow{(Q_{X,X}^{(2)})^{-1}} & Q(X) \times_\Lambda Q(X) & \xrightarrow{Q(\psi) \times_\Lambda Q(\psi)} & Q(K) \times_\Lambda Q(K), \end{array}$$

where $\psi : X \rightarrow K$ is the isomorphism given by Lemma 4.3 and $\tilde{\sigma}$ is the morphism defined by (4.5). By Lemma 4.5, it is sufficient to show that this diagram is commutative to prove this theorem. For simplicity, we set

$$\Xi = (Q(\psi) \times_\Lambda Q(\psi)) \circ (Q_{X,X}^{(2)})^{-1}.$$

For all $\lambda, x, y \in \Lambda$ and $i, j \in X_0$, we compute

$$\begin{aligned} \Xi(\lambda, (i, x), (j, y)) &= ((Q(\psi) \times_\Lambda Q(\psi))((\lambda, i, x), (\lambda *_i x, j, y))) \\ &= ((\lambda, i, \lambda *_i x), (\lambda *_i x, j, (\lambda *_i x) *_j y)) \\ &= \lambda \xrightarrow{i} \lambda *_i x \xrightarrow{j} (\lambda *_i x) *_j y. \end{aligned}$$

By the definition of σ and $\tilde{\sigma}$, we have

$$\begin{aligned} \Xi Q(\sigma)(\lambda, (i, x), (j, y)) &= \Xi(\lambda, (i, x) \xrightarrow{\lambda} (j, y), (i, x) \xleftarrow{\lambda} (j, y)) \\ &= \lambda \xrightarrow{i \triangleright j} \langle \lambda, \lambda *_i x, (\lambda *_i x) *_j y \rangle \xrightarrow{i \triangleleft j} (\lambda *_i x) *_j y \\ &= (\tilde{\sigma} \circ \xi)(\lambda, (i, x), (j, y)). \end{aligned}$$

The proof is done. \square

Proof of Theorem 4.2. This theorem is proved by the same way as Theorem 3.8 but by using (4.5) instead of (3.17). \square

5. WEAK BIALGEBRAS ARISING FROM A SOLUTION

5.1. Weak bialgebras arising from a solution. Throughout this section, we work over a fixed field k . By an algebra, we always mean an associative and unital algebra over the field k . The term ‘coalgebra’ is used in a similar manner. A *weak bialgebra* is an algebra endowed with a coalgebra structure $\Delta : B \rightarrow B \otimes_k B$ and $\varepsilon : B \rightarrow k$ such that

$$\begin{aligned}\Delta(ab) &= \Delta(a)\Delta(b), \\ (\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1)) &= 1_{(1)} \otimes 1_{(2)} \otimes 1_{(3)} = (1 \otimes \Delta(1)) \cdot (\Delta(1) \otimes 1), \\ \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c) &= \varepsilon(abc) = \varepsilon(ab_{(1)})\varepsilon(b_{(2)}c)\end{aligned}$$

for all $a, b, c \in B$. Here we have used the Sweedler notation, such as

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \quad \text{and} \quad \Delta(a_{(1)}) \otimes a_{(2)} = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} = a_{(1)} \otimes \Delta(a_{(2)})$$

to express the comultiplication of $a \in B$.

Let Λ be a non-empty set, and let $\mathcal{F} = \text{Map}(\Lambda, k)$ be the algebra of the k -valued functions on Λ . On the one hand, Shibukawa [22] introduced a construction of a bialgebroid over \mathcal{F} ($= \times_{\mathcal{F}}$ -bialgebra in the sense of Sweedler [24] and Takeuchi [25]) from a dynamical Yang-Baxter map (X, σ) satisfying the following condition:

$$(5.1) \quad \begin{aligned} &\text{the map } \sigma(\lambda) \text{ is invertible for all } \lambda \in \Lambda, \text{ the set } X \text{ is finite,} \\ &\text{and the map } \Lambda \rightarrow \Lambda \text{ given by } \lambda \mapsto \lambda \triangleleft x \text{ is bijective for all } x \in X. \end{aligned}$$

On the other hand, Hayashi [8] gave a construction of a weak bialgebra from a certain class of braided object of the category $\mathcal{F}\text{-Bim}$ of \mathcal{F} -bimodules. Thus, when Λ is finite, one can obtain a weak bialgebra by applying his construction to the linearization of the braided quiver

$$(Q, \tilde{\sigma}) := \text{Br}(Q)(X, \sigma).$$

Suppose that Λ is finite. Then \mathcal{F} is a separable algebra and hence a bialgebroid over \mathcal{F} is a weak bialgebra by Schauenburg [18]. Thus we have two constructions of a weak bialgebra from a dynamical Yang-Baxter map satisfying (5.1). The aim of this section is to discuss relations between these two constructions.

5.2. Hayashi’s construction. We fix a non-empty finite set Λ and consider the algebra $\mathcal{F} = \text{Map}(\Lambda, k)$ of k -valued functions on Λ . If Q is a quiver over Λ , then the vector space $W = \text{span}_k(Q)$ spanned by Q is an \mathcal{F} -bimodule by

$$f \cdot a = f(\mathfrak{s}(a))a \quad \text{and} \quad a \cdot f = f(\mathfrak{t}(a))a$$

for $f \in \mathcal{F}$ and $a \in Q$. A *star-triangular face model* is a pair (Q, w) consisting of a finite quiver Q and a morphism $w : W \otimes_{\mathcal{F}} W \rightarrow W \otimes_{\mathcal{F}} W$ of \mathcal{F} -bimodules, where $W = \text{span}_k(Q)$ is the \mathcal{F} -bimodule associated to Q , such that w is invertible and (W, w) is a braided object of $\mathcal{F}\text{-Bim}$. Hayashi [8] introduced a construction of a weak bialgebra $\mathfrak{A}(w)$ from such a model and studied its Hopf closure, that is, a weak Hopf algebra with a certain universal property for $\mathfrak{A}(w)$.

We recall the construction of $\mathfrak{A}(w)$. Let (Q, w) be a star-triangular face model, and set $W = \text{span}_k(Q)$. For a positive integer m , we define $Q^{(m)}$ inductively by

$$Q^{(1)} = Q \quad \text{and} \quad Q^{(m+1)} = Q^{(m)} \times_{\Lambda} Q.$$

An element of $Q^{(m)}$ can be regarded as a path on Q of length m . For simplicity of notation, we set $Q^{(0)} = \Lambda$. An element $\lambda \in Q^{(0)}$ will be regarded as a path on Q of length zero from λ to λ .

It is easy to see that the set $\{a \otimes b \in W \otimes_{\mathcal{F}} W \mid (a, b) \in Q^{(2)}\}$ is a basis of the k -vector space $W \otimes_{\mathcal{F}} W$. We say that a quadruple (a, b, c, d) of elements of Q form a *face* if $\mathfrak{s}(a) = \mathfrak{s}(c)$, $\mathfrak{t}(a) = \mathfrak{s}(b)$, $\mathfrak{t}(c) = \mathfrak{s}(d)$, $\mathfrak{t}(b) = \mathfrak{t}(d)$. Since w is a morphism of \mathcal{F} -bimodules, we have

$$w(a \otimes b) \in \text{span}_k\{c \otimes d \mid c, d \in Q \text{ and } (a, b, c, d) \text{ form a face}\}$$

for all $(a, b) \in Q^{(2)}$. We define $\mathbf{w} \begin{bmatrix} a & \\ c & b \\ & d \end{bmatrix} \in k$ for a face (a, b, c, d) by

$$w(a \otimes b) = \sum_{(c, d) \in Q^{(2)}} \mathbf{w} \begin{bmatrix} a & \\ c & b \\ & d \end{bmatrix} c \otimes d \quad ((a, b) \in Q^{(2)}).$$

For simplicity, we set $\mathbf{w} \begin{bmatrix} a & \\ c & b \\ & d \end{bmatrix} = 0$ unless (a, b, c, d) form a face.

Definition 5.1. The weak bialgebra $\mathfrak{A}(w)$ associated to the star-triangular face model (W, w) is, as a k -algebra, generated by the symbols

$$\mathbf{e} \begin{bmatrix} p \\ q \end{bmatrix} \quad (p, q \in Q^{(m)}, m = 0, 1, 2, \dots)$$

subject to the following relations:

$$(5.2) \quad \sum_{\lambda, \mu \in Q^{(0)}} \mathbf{e} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 1,$$

$$(5.3) \quad \mathbf{e} \begin{bmatrix} p \\ q \end{bmatrix} \cdot \mathbf{e} \begin{bmatrix} p' \\ q' \end{bmatrix} = \delta_{\mathfrak{t}(p), \mathfrak{s}(p')} \delta_{\mathfrak{t}(q), \mathfrak{s}(q')} \mathbf{e} \begin{bmatrix} pp' \\ qq' \end{bmatrix}$$

for $p, q \in Q^{(m)}$ and $p', q' \in Q^{(n)}$ ($m, n = 0, 1, 2, \dots$), and

$$(5.4) \quad \sum_{(x, y) \in Q^{(2)}} \mathbf{w} \begin{bmatrix} x & y \\ a & b \end{bmatrix} \mathbf{e} \begin{bmatrix} x \\ c \end{bmatrix} \mathbf{e} \begin{bmatrix} y \\ d \end{bmatrix} = \sum_{(x, y) \in Q^{(2)}} \mathbf{w} \begin{bmatrix} c & d \\ x & y \end{bmatrix} \mathbf{e} \begin{bmatrix} a \\ x \end{bmatrix} \mathbf{e} \begin{bmatrix} b \\ y \end{bmatrix}$$

for $(a, b), (c, d) \in Q^{(2)}$. The coalgebra structure of $\mathfrak{A}(w)$ is determined by

$$\Delta \left(\mathbf{e} \begin{bmatrix} p \\ q \end{bmatrix} \right) = \sum_{t \in Q^{(m)}} \mathbf{e} \begin{bmatrix} p \\ t \end{bmatrix} \otimes \mathbf{e} \begin{bmatrix} t \\ q \end{bmatrix} \quad \text{and} \quad \varepsilon \left(\mathbf{e} \begin{bmatrix} p \\ q \end{bmatrix} \right) = \delta_{p, q}.$$

for $p, q \in Q^{(m)}$ ($m = 0, 1, 2, \dots$).

5.3. Shibukawa's construction. Let Λ be a non-empty set, and let \mathcal{F} be the algebra of k -valued functions on Λ . Shibukawa [22] introduced a method to construct a bialgebroid over \mathcal{F} from a dynamical Yang-Baxter map satisfying (5.1). To describe his construction in a short way, we recall the following algebraic constructions:

Definition 5.2. Let R be an algebra over k , and let S be a set. The free R -bimodule RSR with basis S is just the free left R^e -module over S , where $R^e = R \otimes_k R^{\text{op}}$. The free R -ring $R\langle S \rangle$ generated by S is the tensor algebra

$$R\langle S \rangle = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \dots$$

of the R -bimodule $M = RSR$.

If we write $(a \otimes_k b)s \in RSR$ for $a, b \in R$ and $X \in S$ as aXb , then every element of $R\langle S \rangle$ is a finite sum of elements of the form $a_1 X_1 a_2 \cdots a_n X_n a_{n+1}$ for some $a_i \in R$ and $X_i \in S$. We note that an element of R and an element of S do not commute in $R\langle S \rangle$, but an element of k ($\subset R$) is central in $R\langle S \rangle$. Thus $R\langle S \rangle$ is an algebra over k . One can define “the R -ring generated by S subject to the relations ...” as the quotient algebra of $R\langle S \rangle$ by the ideal defined by the relations.

Now let (X, σ) be a dynamical Yang-Baxter map satisfying (5.1). To avoid technical difficulty, we only consider the case where Λ is finite. Then \mathcal{F} is a separable algebra with Frobenius system

$$(5.5) \quad t(f) = \sum_{\lambda \in \Lambda} f(\lambda) \quad (f \in \mathcal{F}) \quad \text{and} \quad e = \sum_{\lambda \in \Lambda} \delta_\lambda \otimes \delta_\lambda \in \mathcal{F} \otimes_k \mathcal{F},$$

where $\delta_\lambda \in \mathcal{F}$ is the function defined by $\delta_\lambda(\mu) = \delta_{\lambda\mu}$ for $\mu \in \Lambda$. By the result of Schauenburg [18], we may regard a bialgebroid over \mathcal{F} as a weak bialgebra over k . We also remark that Shibukawa has considered the equation (2.16) in [22] instead of the set-theoretical dynamical Yang-Baxter equation (1.4) considered in this paper. For this reason, we use the notation (1.6) and define

$$\tilde{\sigma} : \Lambda \times X \times X \rightarrow X \times X, \quad (\lambda, x, y) \mapsto (y \xleftarrow{\lambda} x, y \xrightarrow{\lambda} x)$$

so that $\tilde{\sigma}$ satisfies (2.15) and (2.16) (see Remark 2.10). The weak bialgebra $\mathfrak{B}(\sigma)$ introduced in the below is in fact Shibukawa’s bialgebroid $A_{\tilde{\sigma}}$ over \mathcal{F} regarded as a weak bialgebra over k in Schauenburg’s manner.

Definition 5.3. We set $\mathcal{E} := \mathcal{F} \otimes_k \mathcal{F}$ and write $\xi \otimes 1 \in \mathcal{E}$ and $1 \otimes \xi \in \mathcal{E}$ simply by ξ and $\bar{\xi}$, respectively. The weak bialgebra $\mathfrak{B}(\sigma)$ is the \mathcal{E} -ring generated by the formal symbols L_{ab}^+, L_{ab}^- ($a, b \in X$) subject to the relations

$$(5.6) \quad \sum_{b \in X} L_{ab}^+ L_{bc}^- = 1 = \sum_{b \in X} L_{ab}^- L_{bc}^+,$$

$$(5.7) \quad L_{ab}^+ \cdot \xi = (a \triangleright \xi) \cdot L_{ab}^+, \quad L_{ab}^+ \cdot \bar{\xi} = \overline{b \triangleright \xi} \cdot L_{ab}^+,$$

$$(5.8) \quad L_{ab}^- \cdot (b \triangleright \xi) = \xi \cdot L_{ab}^-, \quad L_{ab}^- \cdot \overline{a \triangleright \xi} = \bar{\xi} \cdot L_{ab}^-,$$

$$(5.9) \quad \sum_{x, y \in X} \sigma_{ab}^{xy} L_{xc}^+ L_{yd}^+ = \sum_{x, y \in X} \overline{\sigma_{xy}^{cd}} L_{ax}^+ L_{by}^+$$

for $a, b, c, d \in X$ and $\xi \in \mathcal{F}$, where

$$(a \triangleright \xi)(\lambda) = \xi(\lambda \triangleleft a) \quad (a \in X, \xi \in \mathcal{F}, \lambda \in \Lambda)$$

and $\sigma_{rs}^{pq} \in \mathcal{F}$ for $p, q, r, s \in X$ is defined by

$$(5.10) \quad \sigma_{rs}^{pq}(\lambda) = \begin{cases} 1 & \text{if } \sigma(\lambda, p, q) = (r, s), \\ 0 & \text{otherwise.} \end{cases}$$

The comultiplication $\Delta : \mathfrak{B}(\sigma) \rightarrow \mathfrak{B}(\sigma) \otimes_k \mathfrak{B}(\sigma)$ is determined by

$$\begin{aligned} \Delta(\xi) &= \sum_{\lambda \in \Lambda} \xi \bar{\delta}_\lambda \otimes \delta_\lambda, & \Delta(\bar{\xi}) &= \sum_{\lambda \in \Lambda} \bar{\delta}_\lambda \otimes \bar{\xi} \delta_\lambda, \\ \Delta(L_{ab}^\pm) &= \sum_{c \in X, \lambda \in \Lambda} \bar{\delta}_\lambda L_{ac}^\pm \otimes \delta_\lambda L_{cb}^\pm \end{aligned}$$

for $\xi \in \mathcal{F}$ and $a, b \in X$. To define the counit, we introduce the left action \rightharpoonup of the algebra $\mathfrak{B}(\sigma)$ on the vector space \mathcal{F} defined by

$$\xi \rightharpoonup f = \xi f = \bar{\xi} \rightharpoonup f \quad \text{and} \quad (L_{ab}^{\pm} \rightharpoonup f)(\lambda) = \delta_{ab} f(\lambda \triangleleft a^{\pm 1})$$

for $\xi \in \mathcal{F}$ and $a, b \in X$, where $(-) \triangleleft a^{+1} = (-) \triangleleft a$ and $(-) \triangleleft a^{-1}$ is the inverse of the map $(-) \triangleleft a$. Now the counit $\varepsilon : \mathfrak{B}(\sigma) \rightarrow k$ is defined by

$$\varepsilon(b) = t(b \rightharpoonup 1_{\mathcal{F}}) \quad (b \in \mathfrak{B}(\sigma)),$$

where $t : \mathcal{F} \rightarrow k$ is the Frobenius trace of \mathcal{F} given by (5.5) and $1_{\mathcal{F}} = \sum_{\lambda \in \Lambda} \delta_{\lambda}$ is the unit element of \mathcal{F} .

5.4. Comparison of two constructions. Let Λ be a non-empty finite set, and let (X, σ) be a dynamical Yang-Baxter map satisfying (5.1). On the one hand, we have the weak bialgebra $\mathfrak{B}(\sigma)$ by Shibukawa's construction. On the other hand, we have a weak bialgebra by composing Theorem 2.9 and Hayashi's construction: Namely, by the theorem, we have a braided object $(Q, \tilde{\sigma}) := \text{Br}(\mathbf{Q})(X, \sigma)$ of Quiv_{Λ} . Set $W = \text{span}_k(Q)$. The map $\tilde{\sigma}$ induces a linear map

$$w_{\sigma} : W \otimes_{\mathcal{F}} W \rightarrow W \otimes_{\mathcal{F}} W, \quad w_{\sigma}(a \otimes b) = c \otimes d \quad ((a, b) \in Q^2, (c, d) = \tilde{\sigma}(a, b)),$$

which gives rise to a star-triangular face model (Q, w_{σ}) . We thus obtain a weak bialgebra $\mathfrak{A}(w_{\sigma})$ from the braided object (X, σ) . The main result of this section is the following relation between these two weak bialgebras:

Theorem 5.4. *For an integer $m \geq 0$ and two elements*

$$(5.11) \quad p = ((\lambda_1, x_1), \dots, (\lambda_m, x_m)) \quad \text{and} \quad q = ((\mu_1, y_1), \dots, (\mu_m, y_m))$$

of $Q^{(m)}$, we set

$$(5.12) \quad \phi \left(\mathbf{e} \begin{bmatrix} p \\ q \end{bmatrix} \right) = \delta_{\lambda_1} \bar{\delta}_{\mu_1} L_{x_1 y_1}^+ \cdots L_{x_m y_m}^+.$$

Then ϕ extends to a morphism $\phi : \mathfrak{A}(w_{\sigma}) \rightarrow \mathfrak{B}(\sigma)$ of weak bialgebras.

Proof. We shall check that ϕ preserves the relations (5.2)–(5.4). It is easy to see that ϕ preserves (5.2). By (5.1), (5.7) and (5.8), we have

$$\delta_{\lambda} L_{xy}^+ = (x \triangleright \delta_{\lambda \triangleleft x}) L_{xy}^+ = L_{xy}^+ \delta_{\lambda \triangleleft x} \quad \text{and} \quad \bar{\delta}_{\mu} L_{xy}^+ = L_{xy}^+ \bar{\delta}_{\mu \triangleleft x}$$

for all $x, y \in X$ and $\lambda, \mu \in \Lambda$. By using these equations and the definition of the quiver $Q = \mathbf{Q}(X)$, one can verify that ϕ preserves (5.3).

We now check that ϕ preserves (5.4). For this purpose, we determine the symbol $\mathbf{w}[\cdots]$ associated to the star-triangular face model (Q, w_{σ}) arising from (X, σ) . Suppose that $((\lambda, a), (\lambda', b))$ and $((\mu, c), (\mu', d))$ are elements of $Q^{(2)}$. By using $\sigma_{cd}^{ab} \in \mathcal{F}$ defined by (5.10), the symbol $\mathbf{w}[\cdots]$ is expressed as follows:

$$\mathbf{w} \left[\begin{array}{cc} (\lambda, a) & (\lambda', b) \\ (\mu, c) & (\mu', d) \end{array} \right] = \sigma_{cd}^{ab}(\lambda) \delta_{\lambda, \mu}.$$

Thus we compute:

$$\begin{aligned}
& \sum_{((\lambda, x), (\lambda', y)) \in Q^2} \mathbf{w} \left[\begin{matrix} (\lambda, x) \\ (\mu, a) \end{matrix} \begin{matrix} (\lambda', y) \\ (\mu', b) \end{matrix} \right] \phi \left(\mathbf{e} \left[\begin{matrix} (\lambda, x) \\ (\nu, c) \end{matrix} \right] \right) \phi \left(\mathbf{e} \left[\begin{matrix} (\lambda', y) \\ (\nu', d) \end{matrix} \right] \right) \\
&= \sum_{\lambda \in \Lambda, x, y \in X} \sigma_{ab}^{xy}(\lambda) \delta_\mu(\lambda) \delta_\lambda \bar{\delta}_\gamma L_{xc}^+ L_{yd}^+ = \sum_{x, y \in X} \sigma_{ab}^{xy} \delta_\mu \bar{\delta}_\gamma L_{xc}^+ L_{yd}^+ \\
&= \delta_\mu \bar{\delta}_\gamma \cdot (\text{the left-hand side of (5.9)}), \\
& \sum_{((\lambda, x), (\lambda', y)) \in Q^2} \mathbf{w} \left[\begin{matrix} (\nu, c) \\ (\lambda, x) \end{matrix} \begin{matrix} (\nu', d) \\ (\lambda', y) \end{matrix} \right] \phi \left(\mathbf{e} \left[\begin{matrix} (\mu, a) \\ (\lambda, x) \end{matrix} \right] \right) \phi \left(\mathbf{e} \left[\begin{matrix} (\mu', b) \\ (\lambda', y) \end{matrix} \right] \right) \\
&= \sum_{\lambda \in \Lambda, x, y \in X} \sigma_{xy}^{cd}(\lambda) \delta_\nu(\lambda) \delta_\mu \bar{\delta}_\lambda L_{ax}^+ L_{by}^+ = \sum_{\lambda \in \Lambda, x, y \in X} \overline{\sigma_{xy}^{cd}} \delta_\mu \bar{\delta}_\nu L_{ax}^+ L_{by}^+ \\
&= \delta_\mu \bar{\delta}_\gamma \cdot (\text{the right-hand side of (5.9)}).
\end{aligned}$$

Namely, ϕ preserves (5.4). Hence ϕ extends to a morphism $\phi : \mathfrak{A}(w_\sigma) \rightarrow \mathfrak{B}(\sigma)$ of algebras over k .

To complete the proof, we check that the algebra map ϕ preserves the coalgebra structure of $\mathfrak{A}(w_\sigma)$. Since the comultiplication of a weak bialgebra is multiplicative, it is sufficient to show

$$\Delta \phi \left(\mathbf{e} \left[\begin{matrix} (\lambda, x) \\ (\mu, y) \end{matrix} \right] \right) = (\phi \otimes \phi) \Delta \left(\mathbf{e} \left[\begin{matrix} (\lambda, x) \\ (\mu, y) \end{matrix} \right] \right)$$

for all $\lambda, \mu \in \Lambda$ and $x, y \in X$ to show that ϕ preserves the comultiplication. This is verified as follows:

$$\begin{aligned}
\Delta \phi \left(\mathbf{e} \left[\begin{matrix} (\lambda, x) \\ (\mu, y) \end{matrix} \right] \right) &= \sum_{\substack{\nu_1, \nu_2, \nu_3 \in \Lambda, \\ c \in X}} (\delta_\lambda \bar{\delta}_{\nu_1} \otimes \delta_{\nu_1}) \cdot (\bar{\delta}_{\nu_2} \otimes \bar{\delta}_\mu \delta_{\nu_2}) \cdot (\bar{\delta}_{\nu_3} L_{xc}^+ \otimes \delta_{\nu_3} L_{cy}^+) \\
&= \sum_{\nu \in \Lambda, c \in X} \delta_\lambda \bar{\delta}_\nu L_{xc}^+ \otimes \bar{\delta}_\mu \delta_\nu L_{cy}^+ \\
&= \sum_{\nu \in \Lambda, c \in X} \phi \left(\mathbf{e} \left[\begin{matrix} (\lambda, x) \\ (\mu, c) \end{matrix} \right] \right) \otimes \phi \left(\mathbf{e} \left[\begin{matrix} (\nu, c) \\ (\mu, y) \end{matrix} \right] \right) \\
&= (\phi \otimes \phi) \Delta \left(\mathbf{e} \left[\begin{matrix} (\lambda, x) \\ (\mu, y) \end{matrix} \right] \right).
\end{aligned}$$

Finally, we check that ϕ preserves the counit. Let p and q be paths on Q of length m as in (5.11). Then, by the definition of the quiver Q , we have

$$(5.13) \quad \lambda_{i+1} = \lambda_i \triangleleft x_i \quad \text{and} \quad \mu_{i+1} = \mu_i \triangleleft y_i \quad (i = 1, \dots, m-1).$$

Since $a \triangleright 1_{\mathcal{F}} = 1_{\mathcal{F}}$ for all $a \in X$, we have

$$\begin{aligned}
\phi \left(\mathbf{e} \left[\begin{matrix} p \\ q \end{matrix} \right] \right) &\rightharpoonup 1_{\mathcal{F}} = \left(\delta_{\lambda_1} \bar{\delta}_{\mu_1} L_{x_1 y_1}^+ \cdots L_{x_m y_m}^+ \right) \rightharpoonup 1_{\mathcal{F}} \\
&= \delta_{x_1, y_1} \delta_{x_2, y_2} \cdots \delta_{x_m, y_m} \delta_{\lambda_1} \delta_{\mu_1},
\end{aligned}$$

where \rightharpoonup is the action of $\mathfrak{B}(\sigma)$ on the vector space \mathcal{F} used in the definition of the counit of $\mathfrak{B}(\sigma)$. Hence,

$$\varepsilon\phi\left(\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\right) = \delta_{x_1, y_1} \delta_{x_2, y_2} \cdots \delta_{x_m, y_m} \delta_{\lambda_1, \mu_1}.$$

In view of (5.13), the right-hand side of this equation is 1 if $p = q$ and 0 otherwise. Thus we conclude that $\varepsilon\phi(a) = \varepsilon(a)$ for all $a \in \mathfrak{A}(w_\sigma)$. \square

Remark 5.5. (1) Let $\mathfrak{B}^+(\sigma)$ be the \mathcal{E} -ring generated by L_{ab}^+ ($a, b \in X$) subject to the relations (5.7) and (5.9). The algebra $\mathfrak{B}^+(\sigma)$ is a weak bialgebra by the coalgebra structure defined in the same way as $\mathfrak{B}(\sigma)$. The weak bialgebra $\mathfrak{A}(w_\sigma)$ is isomorphic to $\mathfrak{B}^+(\sigma)$. Indeed, the algebra map $\mathfrak{A}(w_\sigma) \rightarrow \mathfrak{B}^+(\sigma)$ defined by the same formula as (5.12) has the inverse determined by

$$L_{ab}^+ \mapsto \sum_{\lambda, \mu \in \Lambda} \mathbf{e}\begin{bmatrix} (\lambda, a) \\ (\mu, b) \end{bmatrix}, \quad \xi_1 \bar{\xi}_2 \mapsto \sum_{\lambda, \mu \in Q^{(0)}} \xi_1(\lambda) \xi_2(\mu) \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$$

for $a, b \in X$ and $\xi_1, \xi_2 \in \mathcal{F}$. The algebra $\mathfrak{B}(\sigma)$ has extra generators L_{ab}^- and hence is not isomorphic to $\mathfrak{A}(w_\sigma)$ in general.

(2) Let (W, w) be a star-triangular face model. Hayashi [8] showed that the weak bialgebra $\mathfrak{A}(w)$ has a Hopf closure $\mathfrak{H}(w)$ under the assumption that w is *closable* [8, Section 3]. We recall that, as an algebra, $\mathfrak{H}(w)$ is generated by

$$\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}, \quad \mathbf{e}\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} \quad (p, q \in Q^{(2)})$$

subject to the relations like (5.2)–(5.4). Notably, Hayashi introduced an extension \mathbf{w}_{LD} of \mathbf{w} , called the Lyubashenko double, and showed that the equation

$$(5.14) \quad \sum_{(x, y) \in Q^{(2)}} \mathbf{w}_{\text{LD}} \begin{bmatrix} \bar{x} \\ \bar{a} \end{bmatrix} \begin{bmatrix} \bar{y} \\ \bar{b} \end{bmatrix} \mathbf{e}\begin{bmatrix} \bar{x} \\ \bar{c} \end{bmatrix} \mathbf{e}\begin{bmatrix} \bar{y} \\ \bar{d} \end{bmatrix} = \sum_{(x, y) \in Q^{(2)}} \mathbf{w}_{\text{LD}} \begin{bmatrix} \bar{c} \\ \bar{x} \end{bmatrix} \begin{bmatrix} \bar{d} \\ \bar{y} \end{bmatrix} \mathbf{e}\begin{bmatrix} \bar{a} \\ \bar{x} \end{bmatrix} \mathbf{e}\begin{bmatrix} \bar{b} \\ \bar{y} \end{bmatrix}$$

and its variants hold in $\mathfrak{H}(w)$ [8, Proposition 5.5]. Now we consider the case where (Q, w_σ) is a star-triangular face model arising from a dynamical Yang-Baxter map (X, σ) satisfying (5.1). Shibukawa observed that $\mathfrak{B}(\sigma)$ has an antipode (and thus it is a weak Hopf algebra by [18]) when σ is *rigid* [22, Section 3]. Since we do not know any relations between Hayashi's closability and Shibukawa's rigidity, we assume that, for the time being, w_σ is closable and σ is rigid. Then, by the universal property of the Hopf closure, one can extend the map $\phi : \mathfrak{A}(w_\sigma) \rightarrow \mathfrak{B}(\sigma)$ of the above theorem to a map $\mathfrak{H}(w_\sigma) \rightarrow \mathfrak{B}(\sigma)$ of weak Hopf algebras. This map does not seem to be an isomorphism in general. The difficulty is that $\mathfrak{B}(\sigma)$ has no defining relations corresponding to (5.14) and its variants.

REFERENCES

- [1] Nicolás Andruskiewitsch. On the quiver-theoretical quantum Yang-Baxter equation. *Selecta Math. (N.S.)*, 11(2):203–246, 2005. With an appendix by Mitsuhiro Takeuchi.
- [2] A. H. Clifford. Bands of semigroups. *Proc. Amer. Math. Soc.*, 5:499–504, 1954.
- [3] V. G. Drinfel'd. On some unsolved problems in quantum group theory. In *Quantum groups (Leningrad, 1990)*, volume 1510 of *Lecture Notes in Math.*, pages 1–8. Springer, Berlin, 1992.

- [4] Pavel Etingof and Frédéric Latour. *The dynamical Yang-Baxter equation, representation theory, and quantum integrable systems*, volume 29 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2005.
- [5] Pavel Etingof and Alexander Varchenko. Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Comm. Math. Phys.*, 196(3):591–640, 1998.
- [6] Giovanni Felder. Conformal field theory and integrable systems associated to elliptic curves. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1247–1255. Birkhäuser, Basel, 1995.
- [7] J.-L. Gervais and A. Neveu. Novel triangle relation and absence of tachyons in Liouville string field theory. *Nuclear Phys. B*, 238(1):125–141, 1984.
- [8] Takahiro Hayashi. Quantum groups and quantum semigroups. *J. Algebra*, 204(1):225–254, 1998.
- [9] Noriaki Kamiya and Youichi Shibukawa. Dynamical Yang-Baxter maps associated with homogeneous pre-systems. *J. Gen. Lie Theory Appl.*, 5:Art. ID G110106, 9, 2011.
- [10] V. Lebed. Cohomology of idempotent braidings, with applications to factorizable monoids. [arXiv:160708081](https://arxiv.org/abs/160708081).
- [11] Victoria Lebed. Categorical aspects of virtuality and self-distributivity. *J. Knot Theory Ramifications*, 22(9):1350045, 32, 2013.
- [12] Victoria Lebed. Homologies of algebraic structures via braidings and quantum shuffles. *J. Algebra*, 391:152–192, 2013.
- [13] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
- [14] Diogo Kendy Matsumoto. Dynamical braces and dynamical Yang-Baxter maps. *J. Pure Appl. Algebra*, 217(2):195–206, 2013.
- [15] Diogo Kendy Matsumoto and Youichi Shibukawa. Quantum Yang-Baxter equation, braided semigroups, and dynamical Yang-Baxter maps. *Tokyo J. Math.*, 38(1):227–237, 2015.
- [16] J. B. McGuire. Study of exactly soluble one-dimensional N -body problems. *J. Mathematical Phys.*, 5:622–636, 1964.
- [17] Wolfgang Rump. Dynamical groups and braces. *J. Algebra Appl.*, 15(7):1650135, 31, 2016.
- [18] Peter Schauenburg. Weak Hopf algebras and quantum groupoids. In *Noncommutative geometry and quantum groups (Warsaw, 2001)*, volume 61 of *Banach Center Publ.*, pages 171–188. Polish Acad. Sci. Inst. Math., Warsaw, 2003.
- [19] Y. Shibukawa. Survey on dynamical Yang-Baxter maps. In *Noncommutative structures in mathematics and physics*, pages 239–244. K. Vlaam. Acad. Belgie Wet. Kunsten (KVAB), Brussels, 2010.
- [20] Youichi Shibukawa. Dynamical Yang-Baxter maps. *Int. Math. Res. Not.*, (36):2199–2221, 2005.
- [21] Youichi Shibukawa. Dynamical Yang-Baxter maps with an invariance condition. *Publ. Res. Inst. Math. Sci.*, 43(4):1157–1182, 2007.
- [22] Youichi Shibukawa. Hopf algebroids and rigid tensor categories associated with dynamical Yang-Baxter maps. *J. Algebra*, 449:408–445, 2016.

- [23] Youichi Shibukawa and Mitsuhiro Takeuchi. FRT construction for dynamical Yang-Baxter maps. *J. Algebra*, 323(6):1698–1728, 2010.
- [24] Moss E. Sweedler. Groups of simple algebras. *Inst. Hautes Études Sci. Publ. Math.*, (44):79–189, 1974.
- [25] Mitsuhiro Takeuchi. Groups of algebras over $A \otimes \overline{A}$. *J. Math. Soc. Japan*, 29(3):459–492, 1977.
- [26] C. N. Yang. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.*, 19:1312–1315, 1967.